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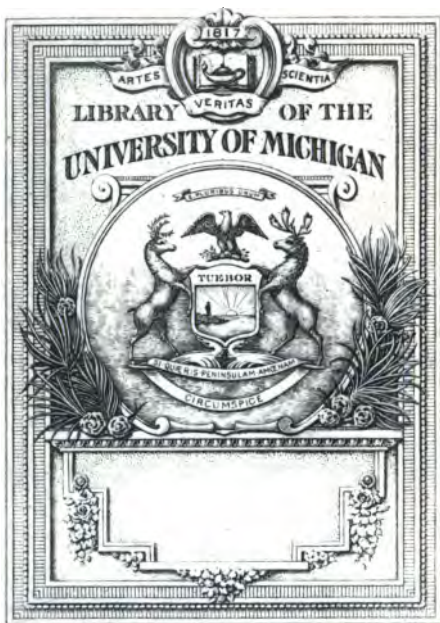
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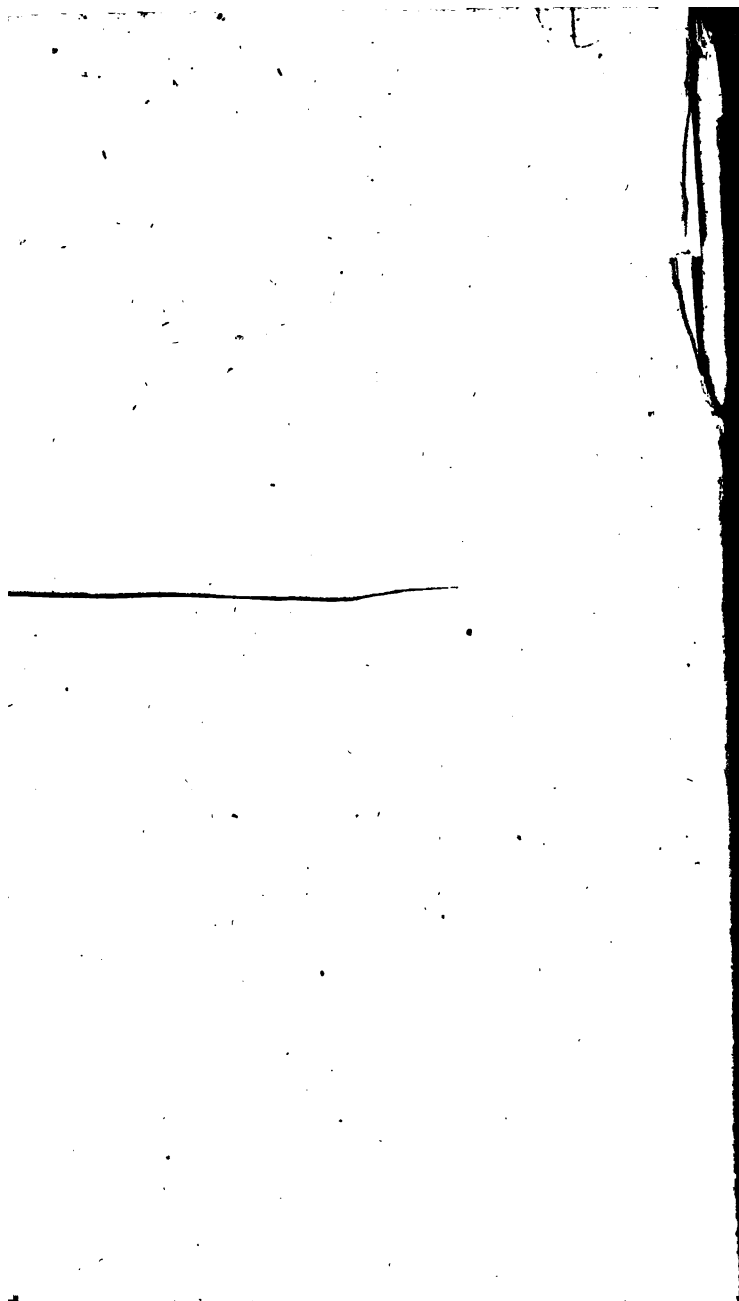


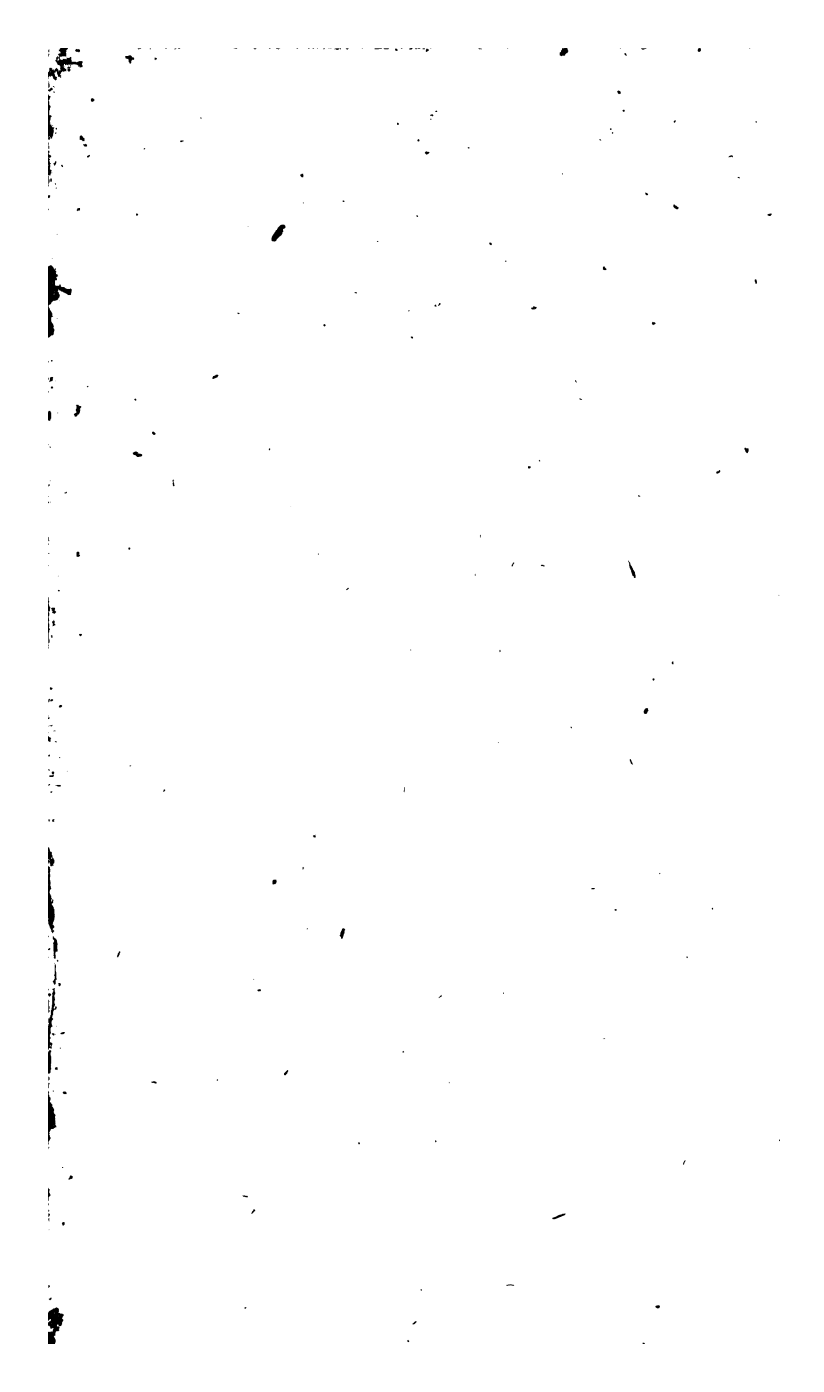
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THE
MATHEMATICAL
REPOSITORY.

BY ^{Thomas} T. LEYBOURN.

VOL. II.

The Mathematics are an universal science, which connects
all the rest, and displays them in their happiest relations.

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TO THE

REV. NEVIL MASKELYNE, D.D. F.R.S.,

ASTRONOMER ROYAL,

WHO,

BY A LONG AND

Indefatigable Application

OF HIS

Eminent Talents,

HAS DONE

EQUAL HONOUR AND SERVICE

TO HIS COUNTRY

AND

TO SCIENCE,

THIS VOLUME

OF THE

MATHEMATICAL REPOSITORY

IS

RESPECTFULLY DEDICATED

BY

THE EDITOR.

03-27-40 mEC



PREFACE.

THE Editor will not attempt to write an eulogium upon the Mathematical Sciences; because he is conscious that in their present advanced state, this would be, as it were,

——— “To add another hue unto the Rainbow;
“or with taper light to seek the beauteous Eye of Heav’n to
“garnish.”

That the greatest and wisest philosophers of which ancient or modern times can boast, have also been eminent Mathematicians, is a circumstance, which of itself is enough to induce all who aim at the enlargement of human knowledge and the consequent amelioration of society, to walk in that path, which has been trodden successfully from the days of THALES and PYTHAGORAS to those of BARROW and NEWTON; and which is still persevered in with honour and advantage. For, not only are the direct and immediate benefits of Mathematics obvious; but their incidental and collateral advantages are also very manifest; as has been often observed, and is expressed with peculiar beauty by the great LORD BACON, in his *Advancement of Learning*: “Pure Mathematics,” saith this noble precursor of Newton, “do
“remedy and cure many defects in the wit, and faculties
“intellectual. For if the wit be dull, they sharpen it; if
“too wandering, they fix it; if too inherent in the sense,
“they abstract it. So that as tennis is a game of no use in
“itself, but of great use in respect it maketh a quick eye,
“and a body ready to put itself in all positions; so, in the
“Mathematics, that use which is collateral and intervenient,
“is no less worthy than that which is principal and intended.”

It is, therefore, not without sufficient reason that mathematical studies have been pursued with great avidity, in almost all ages; and it reflects considerable honour upon the present age, that such avidity is not diminishing, but evidently increasing.

With respect to periodical publications like the present, the experience of the eighteenth century has forcibly demonstrated their utility: indeed the Editor has reason to think there are scarcely any mathematicians of eminence in this country, who will not readily acknowledge, that in one part or other of their scientific course, they have been assisted by such

such works. Under this view of their beneficial tendency, the Editor trusts it will not be thought presumptuous in him to say that they are to the British a kind of *Gymnasium*, in some respects more honourable than the *Gymnasium* of the Grecians; for the exercises among the Grecians frequently inflamed the boisterous passions and stimulated to works of destruction; but in our *Gymnasium* the exercises tend only to promote a generous emulation who shall most expand the sphere of intellect, and who shall gain the deepest insight into the Laws of Nature, and soothing arts of peace.

The Editor of the *Mathematical Repository*, hopes that the publication he has had the honour of issuing into the world, will be found to have yielded its share of amusement and instruction: the nature of his plan must be now thoroughly understood,—for the Contents of each Number, together with the brief remarks which have accompanied, have undoubtedly rendered this clear. The ability and zeal of many of his Correspondents, for the advancement of science, deserves his warmest thanks. And he ventures to hope that his exertions, assisted by their talents and genius, will together make the *Repository* a *depot* of valuable matter; such as may be had recourse to with pleasure and advantage, by every friend to knowledge, and every promoter of science.

May, 1801.

CONTENTS

OF THE SECOND VOLUME.

ART.	PAGE
I. LANDEN on the Ellipsis and Hyperbola, concluded	2
II. Geometrical Propositions, with XXXVII. Problems for Exercise. By Mr. <i>John Lowry</i>	6
III. XIV. XXV. XXXVII. XLV. and LII. several Props. from <i>Lawson</i> on the Ancient Analysis; to be demonstrated	15, 78, 165, 302, 378, and 465
IV. and XV. Several Props. from Dr. <i>Stewart's</i> General Theorems	16 and 79
V. On the resolution of Indeterminate Problems. By <i>John Leslie, A. M.</i> concluded	17
VI. XXXI. Useful Propositions in Geometry. By Mr. <i>M. A. Harrison</i>	23 and 234
VII. Demonstrations of Mr. <i>Keith's</i> method of constructing an Azimuth	26
VIII. Demonstrations of Mr. <i>Emerson's</i> Forms of Fluents. By Rev. <i>L. Evans</i>	27
IX. Answers to Mathematical Questions, 69 to 88, proposed in Art. XLIII. Vol. I.	34
X. New Questions, 110 to 130, answered in Art. XXVII.	57
XI. Mr. <i>Howard's</i> Defence of his Geometry, in Reply to Art. LX. Vol. I.	61
XII. A Letter from Mr. <i>Freud</i> , on the Negative Sine	65
XIII. XXIII. XLII. Demonstrations to <i>Lawson's</i> Propositions	71, 158, and 316
XVI. On the Description of Parabolic Trajectories. By Mr. <i>William Wallace</i>	80
XVII. Solutions to Questions, 89 to 109, proposed in Art. LVIII. Vol. I.	83
XVIII. New Questions, 131 to 156, answered in Art. XXXIII.	106
XIX. Mathematical Lucubrations. By Mr. <i>Wm. Wallace</i>	111
XX. Extracts from a Paper on the Trigonometrical Tables of the Brahmans. By <i>Professor Playfair</i> , F.R.S. Edin.	119
XXI. XXXII. and XLIX. Investigations for determining the times of Vibration of Watch Balances, by <i>George Atwood</i> , Esq. F.R.S.	125, 237, and 401
XXII. Demonstrations to Dr. <i>Stewart's</i> Propositions	136
XXIV. A. B. on Fluents, with an investigation	160
	XXVI.

ART.	PAGE
XXVI. An old Problem on Twilight, with a new Solution. By Mr. <i>John Surtees</i>	166
XXVII. Answers to Questions, 110 to 130, proposed in Art. X.	168
XXVIII. New Questions, 157 to 182, answered in Art. XLIII.	199
XXIX. Hints relative to Friction in Mechanics. By Mr. <i>Reuben Burrow</i> .	204
XXX. and XLVI. Four Propositions on the Sun's Distance from the Earth, by Mr. <i>Dawson</i>	220 and 385
XXXIII. Solutions to Questions, 131 to 156, proposed in Art. XVIII.	247
XXXIV. New Questions, 183 to 210, answered in	291
XXXV. Method of discovering the number of negative and im- possible Roots in any equation. By Mr. <i>Freud</i>	297
XXXVI. Remarks on Mr. Landen's Correction of Emerson's solution of the Tides. By Mr. <i>Benj. Gompertz</i>	300
XXXVIII. Lowry on Pressure	303
XXXIX. A Problem, in addition to the Lemmas of Sir Isaac Newton. By Mr. <i>Robert Wallace</i> .	308
XL. Correction of an error in Art. XXXV. By Mr. <i>Freud</i>	309
XLI. Geometrical Sections. By Mr. <i>Lowry</i> .	312
XLIII. Solutions to Questions, 157 to 182, proposed in Art. XXVIII.	323
XLIV. New Questions, 211 to 240, answered in Vol. III.	371
XLVII. Improved Solution to Question 109	393
XLVIII. Mr. Cunliffe's new method of finding Fluents	394
L. Solutions to Questions, 183 to 210, proposed in Art XXXIV.	414
LI. New Questions, 241 to 270, answered in Vol. III.	459

THE
MATHEMATICAL REPOSITORY.

ARTICLE I.

Landen, on the Ellipsis and Hyperbola.

(Continued from p. 173, vol. 1.)

12. **B**Y substituting $a-b$, $2\sqrt{ab}$, and $(a-b)^2-t^2$ for m , n and mz respectively, in art. 1, it appears that, if (in the hyperbola, fig. 34, plate 11,) the semi-transverse axis AC be $= a-b$, the semi-conjugate $= 2\sqrt{ab}$, and the perpendicular $CP = \sqrt{(a-b)^2-t^2}$; the difference $(DP-AD)$ between the tangent DP and the arch AD will be equal to the fluent of $t \sqrt{((a-b)^2-t^2) \div ((a+b)^2-t^2)}$.

13. It is well known, that, in any ellipsis whose semi-transverse axis is h , and semi-conjugate n , if x be the abscissa, measured from the centre upon the semi-transverse axis, and Q the arch between the conjugate axis and the ordinate corresponding to x , $\frac{\sqrt{(h^2-gx^2) \div (h^2-x^2)}}{(h^2-n^2) \div h^2}$ will be $= Q$, g being $= \frac{h^2-n^2}{h^2}$.

Hence, by substituting $a+b$, $2\sqrt{ab}$, and $t(a+b) \div (a-b)$ for h , n and x respectively; it appears, that, in the ellipsis aed (fig. 26, pl. 1.) whose semi-transverse axis cd is $= a+b$, semi-conjugate $ca = 2\sqrt{ab}$, and abscissa cb (corresponding to the ordi-

nate $be = t(a+b) \div (a-b)$, the arc ae (denoted by Q) will be equal to the fluent of $t \sqrt{((a+b)^2 - t^2)} \div ((a-b)^2 - t^2)$.

14. In the ellipsis $ac'd$ (fig. 27, pl. 1.) the semi-transverse axis cd being $= a$, the semi-conjugate $ca = b$, and the abscissa cb' (corresponding to the ordinate $b'e'$) $= x$; if $e'p'$ the tangent at e' intercepted by a perpendicular (cp') drawn thereto from the centre c , be denoted by t ; $gx \sqrt{(a^2 - x^2)} \div (a^2 - gx^2)$ (as is well known) will be $= t$, g being $= (a^2 - b^2) \div a^2$.

Whence we have $x^2 = (ag^2 + t^2) \div 2g - \sqrt{(a^2 - b^2)^2 - 2(a^2 + b^2)t^2 + t^4} \div 2g$, from which equation, by taking the fluxions, we have $xx' = tt' \div 2g + ((a^2 + b^2)tt' - t^3t') \div 2g \sqrt{(a^2 - b^2)^2 - 2(a^2 + b^2)t^2 + t^4} = t' \div 2g + ((a^2 + b^2)tt' - t^3t') \div 2g \sqrt{((a-b)^2 - t^2) \cdot ((a+b)^2 - t^2)}$. But \dot{R} , the fluxion of the arc ae , being $= x' \sqrt{(a^2 - gx^2)} \div (a^2 - x^2)$ according to the preceding article, it follows that $gx' \div t$ is $= \dot{R}$. It is obvious therefore that

$$\begin{aligned} \dot{R} & \text{ is } = \frac{t}{2} + \frac{(a^2 + b^2)t' - t^3t'}{2\sqrt{(a-b)^2 - t^2} \cdot ((a+b)^2 - t^2)} \\ & = \frac{t}{2} + \frac{((a-b)^2 - t^2)t'}{4\sqrt{((a-b)^2 - t^2) \cdot ((a+b)^2 - t^2)}} + \frac{((a+b)^2 - t^2)t'}{4\sqrt{((a-b)^2 - t^2) \cdot ((a+b)^2 - t^2)}} \\ & = \frac{1}{2}t + \frac{1}{4} \times \frac{((a-b)^2 - t^2)^{\frac{1}{2}}}{((a+b)^2 - t^2)} \times t' + \frac{1}{4} \times \frac{((a+b)^2 - t^2)^{\frac{1}{2}}}{((a-b)^2 - t^2)} \times t' \end{aligned}$$

From

From whence, by taking the fluents, according art. 12 and 13, we find $R = ae$ (fig. 27, pl. 1.) $= \frac{1}{2}t + \frac{1}{2}(DP - AD)$ (fig. 34, pl. 2.) $+ \frac{1}{2}Q$. Consequently the hyperbolic arc AD is $= DP + Q - 4R + 2t$. Thus, beyond my expectation, I find that the *hyperbola* may in general be rectified by means of two *ellipses*!

If $p'e'$, $p''e''$ be equal tangents to the ellipsis $ae'e''d$; the arc ae' (denoted by R) will (by art. 9.) be equal to the arc de'' + the tangent $p''e''$ or $p'e'$ (denoted by t). Therefore, substituting for t its value found by this last equation, it appears that AD is $= DP + Q - 2R - 2 \times de'' = DP + Q + 2 \times e'e'' - 2E''$, and $DP - AD = 2E'' - 2 \times e'e'' - Q$; E'' being put for the quadrantal arc, ad . It is observable that when t is $= a - b$, e'' (by art. 5.) coincides with e' , and $e'e''$ is $= 0$; Q (by art. 13.) $=$ the quadrantal arc (ad); and (by art. 12.) DP and AD both become infinite.

Consequently writing E for that quadrantal arc (ad), and L the *limit* of the difference $DP - AD$, whilst the point of contact (D) is supposed to be carried to an infinite distance from the vertex A of the hyperbola, we find $2E'' - E = L$.

15. Exterminating a , b , and t , by means of the equations $a - b = m$, $2\sqrt{ab} = n$ and $(a - b)^2 - t^2 = mz$; and writing f for $(m^2 - n^2) \div 2m$, as in art. 1.

it appears that (V) the fluent of $\frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}z \div \sqrt{n^2 + 2fz - z^2}$ is $= 2 \times e'e'' - de$ generated whilst z increases from 0. Moreover the fluent of $\frac{1}{2}m^{\frac{1}{2}}z^{-\frac{1}{2}}z$

$\sqrt{(n^2 \div m + z) \div (m - z)}$, or its equal $(\frac{1}{2}m^{-\frac{1}{2}}n^2z^{-\frac{1}{2}}$

$z + \frac{1}{2}m^{\frac{1}{2}}z^{\frac{1}{2}}z) \div \sqrt{n^2 + 2fz - z^2}$ being $= de$, as observed in art. 7, we find by subtraction (W) the

ARTICLE II.

GEOMETRICAL PROPOSITIONS.

By Geometricus. (Fig. 201, Pl. 14.)

LET ABC be a triangle inscribed in the circle $ABFC$, whose centre is R , and let HMN be a circle, inscribed in the said triangle, whose centre is O ; also let IQI' , XqY , $xq'y$, be circles, each touching a side of the triangle, and the continuation of the other two, K , W , w , being their respective centres. Draw OK , OW , Ow , which will bisect the arcs AB , AC , BC , in L , l , d , and meet the sides of the triangle in the angular points C , B , A , respectively; also demit the perpendiculars KI' , KI , KQ , OH , OM , ON , WX , WY , Wq , wx , wy , wq' , LT , dt , lt' , CP , Bp , Ap' ; join LB , LA , BR , AR , BO , AO , BF , BK , AK , AW , CW , Bw , Cw , and draw $LERF$, $leRf$, $dhRk$, which will be perpendicular to AB , AC , BC , and will also bisect them in the points E , e , h .

Now, if AB be the base of the triangle ABC , the following properties are well known, viz.

1. $CI = CI' = BX = BY = Ax = Ay = \frac{1}{2}$ the perimeter.

2. $CM = CN = \frac{1}{2}(AC + BC - AB) =$ half the difference between the sum of the sides and the base.

3. $CT =$ half the sum of the sides AC , CB .

4. $BT = QE = EH =$ half their difference.

5. $AB = IM = I'N$.

6. $2Bt' = AB + BC$, and $2At' = AB + AC = 2Bt' - BC = 2Ct' = 2eM$.

7. $2At$

7. $\angle At = \angle AB + \angle AC$, and $\angle Bt = \angle AB \oslash \angle AC = q/h = hN$.

8. $Xy = \angle AC + \angle BC$; $Ix = \angle AB + \angle BC$; and $I'Y = \angle AB + \angle AC$.

9. $AX = By = CM$.

10. $Al = Cx = BH = AQ = Mm = BN$.

11. $BI' = Cy = AH = BQ = Im = AM$.

12. $Hy = Mx = BC$, and $HX = NY = AC$.

PROP. I.

A circle described about the centre L, with the distance BL, or AL, will pass through the points O and K.

For, the $\angle LAB = \angle ACL$, and $\angle BAO = \angle OAC$; therefore $\angle LAO = \angle LOA$, and consequently $LA = LO$; therefore the circle passes through the point O.

Again, the $\triangle IKA = \triangle QKA$; therefore $\angle KAI = \angle KAB$; hence, the $\angle KAO = \angle KAI + \angle OAC =$ a right angle; consequently the circle passes through the point K.

Cor. 1. A circle described about the centre l, with the distance lA, or lC, will pass through the points O and W.

Cor. 2. A circle described about the centre d, with the distance dB, or dC, will pass through the points O and w.

Cor. 3. The right lines joining the points KW, Ww, and Kw, will pass through the points A, k; F, C; and B, f, respectively.

Cor. 4. $NT = TI' =$ half the base AB.

Cor. 5. $TU = TB =$ half the difference of the sides.

Cor. 6. The rect. $ACB =$ rect. $UCB =$ rect. OCK .

Cor. 7. The rect. $BAC =$ rect. OAw .

Cor. 8. The rect. $ABC =$ rect. OBW .

PROP.

PROP. II.

The distance of the centre of the triangle ABC, is to the radius of the circumscribing circle, as the radius of the inscribed circle, to the distance of the centre of the circle touching the three sides of the triangle from the vertex of the triangle.

The triangles OMN, ONI, SEF, are similar;
 $OM : ON :: ON : OM$;
 $ON : NI :: NI : IC$.

$$OM : ON :: SE : EI :: IC : IK.$$

$$EI = IK - OM.$$

$$OM : ON :: SE : IK - OM.$$

$$OM : SE :: IK - OM : EI.$$

$$OM : SE :: IK - OM : IK - EI.$$

$$OM : SE :: IK - EI : EI.$$

$$IC^2 + CS^2 + AS^2 = BE^2 + EF^2.$$

$$OM : ON :: OM + EI : AH.$$

$$IC^2 + CS^2 = AS^2 - OM^2 (OM + EI).$$

$$EI : IK :: IK - EI : AH.$$

$$IC^2 + CS^2 = AB^2 - IK^2 (IK - EI).$$

PROP. III.

The segment FI, is to the diameter FL, so is the radius of the inscribed circle, to the distance of the centre of that circle from the vertex of the triangle; so is the radius of the circle touching the three sides of the triangle, to the distance of the centre of the circle touching the three sides from the vertex of the triangle.

to the square of the distance of its centre from the vertex of the triangle.

For, by sim. Δ 's $BE : EF :: OM : CM$,
 and $EL : BE :: OM : CM$;
 therefore $EL : EF :: OM^2 : CM^2$;
 and theref. $EL : FL :: OM^2 : OM^2 + CM^2 = OG^2$.

In the same way it is shewn that

$$EL : FL :: IK^2 : KC^2.$$

Cor. 1. As $OC \cdot CK : OM \cdot IK :: FL : EL$.

Cor. 2. As $OC \cdot CK : OC^2 :: CI : CM$.

Cor. 3. Join II' , XY , xy , and draw Mn perpendicular to CK . Then $OC \cdot IZ = OB \cdot XZ' = OA \cdot xZ'' = IK \cdot CM = WX \cdot BH = wy \cdot AH = KC \cdot Mn = OM \cdot IC =$ the area of the ΔABC .

Cor. 4. If the perpendiculars IK , WX , and wy , be given, the ratio of AK , AH , and BH , will be given. And if the distances OC , OB , OA , be given, the ratio of IZ , XZ' , and xZ'' , that is, the ratio of the sines of half the angles of the triangle will be given.

Cor. 5. The rect. $BH \cdot HA =$ rect. $IK \cdot OM$.

Cor. 6. As $BE^2 : OM \cdot IK :: EF : PC$.

Cor. 7. As $BE^2 : OM \cdot IK :: EFL : OCK = ACB$.

Cor. 8. As $BE : EE :: IK \cdot OM : \text{area } \Delta ABC$.

PROP. IV.

As the base is to half the difference between the sum of the sides and the base, so is the diameter of the inscribed circle, to the difference between the perpendicular and the diameter of the said circle.

For, $AB + CM : AB :: PC : 2OM$,
 hence $AB : CM :: 2OM : PC - 2OM$.

Cor. 1. As $AB : IC :: 2IK : PC + 2IK$.

Cor. 2. As $AB^2 : 4OM^2 :: FE : PC - 2OM$.

Cor. 3. As $AB^2 : 4IK^2 :: FE : PC + 2IK$.

PROP.

PROP. V.

The distance between the centres of the inscribed and circumscribing circles, is a mean proportional between the radius of the circumscribing circle, and the difference between the said radius and the diameter of the inscribed circle.

Let Rb be drawn \perp to CL, and then $Lb = Cb$;
therefore $LR^2 - RO^2 = Lb^2 - Ob^2 = LOC$;

but the Δ s COM, LFA, are similar;

therefore $LA \cdot OC$ or $LOC = LF \cdot OM$;

wherefore $LR^2 - RO^2 = LF \cdot OM$;

therefore $RO^2 = LR \cdot (LR - 2OM)$.

Cor. 1. $KR^2 = LR \cdot (LR + 2IK)$.

Cor. 2. $WR^2 = LR \cdot (LR + 2WX)$.

Cor. 3. $wR^2 = LR \cdot (LR + 2wy)$.

Cor. 4. $KR^2 - OR^2 = 2LR \cdot (IK + OM) = 2LR \cdot EL \cdot (AC + CB) \div BE$.

The preceding Propositions, though simple in appearance, are nevertheless very extensive in their application. They enable us to give *simple* and *elegant Constructions* to a great variety of curious and difficult Problems in plain Geometry; a few of which I have selected as exercises for young Students in Geometry. Their solutions will reward the trouble of obtaining them, by the number of new Propositions which they will unavoidably suggest, and which I have omitted to exhibit above, as I did not wish to deprive the young Geometrician of the pleasure of investigating them himself.

Prob. I. Given the base, the vertical angle, and either the sum, difference, or rectangle of the
sum

sum of the sides and radius of the inscribed circle, to determine the triangle.

Prob. II. Given the vertical angle, the perimeter, and either the rectangle of the sides, or the area, to construct the triangle.

Prob. III. Given the perpendicular, and the radii of the circumscribing and inscribed circles, to describe the triangle.

Prob. IV. Given the base, the vertical angle, and the difference between the perpendicular and the diameter of the inscribed circle, to determine the triangle.

Prob. V. Given the vertical angle, the sum or difference of the sides, and the sum or difference of the base and radius of the inscribed circle, to determine the triangle.

Prob. VI. Given the vertical angle, the rectangle of the sides, and the rectangle of the radii of the circumscribing and inscribed circles, to construct the triangle.

Prob. VII. Given the vertical angle, the rectangle of the sides, and the sum, difference, or rectangle of the base and radius of the inscribed circle, to construct the triangle.

Prob. VIII. Given the vertical angle, the perimeter, and the difference of the sides, to construct the triangle.

Prob. IX. Given the vertical angle, the rectangle under the sum of the sides and the base, and the sum of the squares of the base, and radius of the inscribed circle, to construct the triangle.

Prob. X. Given two sides, and the radius of the inscribed circle, to construct the triangle.

Prob. XI. Given two sides, and the distance of their intersection from the centre of the inscribed circle, to describe the triangle.

Prob. XII. Given the radius of the inscribed circle

made by its contact with the said circle, and the difference between the base and the diameter of the circumscribing circle, to determine the triangle.

Prob. XIII. Given the vertical angle, the rectangle of the sides, and the rectangle of the base and sum of the sides, to construct the triangle.

Prob. XIV. Given the difference of the angles at the base, the sum of the squares of the sides, and the distance between the centre of the inscribed circle and the vertex, to construct the triangle.

Prob. XV. Given the vertical angle, the radius of the inscribed circle, and the rectangle under the base and sum of the sides, to construct the triangle.

Prob. XVI. Given the vertical angle, either the sum of the sides, or the radius of the inscribed circle, and the rectangle of the segments of the base made by the point of contact with the inscribed circle, to construct the triangle.

Prob. XVII. Given the radii of the circumscribing and inscribed circles, and the nearest distance of the periphery of the latter from the vertex, to describe the triangle.

Prob. XVIII. Given the sum of the sides; and the radii of two circles, each touching a side of the triangle, and the continuation of the base and the other side, to construct the triangle.

Prob. XIX. Given the base, the line bisecting the vertical angle, and the difference between the radius of the inscribed circle and the radius of a circle touching the base, and the continuation of the other two sides, to determine the triangle.

Prob. XX. Given the perimeter, and the radii of two circles, each touching a side of the triangle of

and the continuation of the other two sides, to determine the triangle.

Prob. XXI. Given the sum of the sides, the sum of the radii of the inscribed circle and the circle touching the base, and the continuation of the other two sides, and the line bisecting the base, to determine the triangle.

Prob. XXII. Given the three distances from the centre of the inscribed circle to the angles of the triangle, to describe it.

Prob. XXIII. Given the base, the vertical angle, and the radius of a circle touching the base and the continuation of the other two sides, to describe the triangle.

Prob. XXIV. Given the base and the radii of two circles, each touching a side and the continuation of the other side and base, to construct the triangle.

Prob. XXV. Given the area, the rectangle of the segments of the base made by the point of contact of the inscribed circle, and the line bisecting the vertical angle, to construct the triangle.

Prob. XXVI. Given the vertical angle, the sum of the sides, and the distance between the centres of the circumscribing and inscribed circles, to construct the triangle.

Prob. XXVII. Given the vertical angle, the difference between the perpendicular and the diameter of the inscribed circle, and the distance between the centres of the circumscribing and inscribed circles, to construct the triangle.

Prob. XXVIII. Given the perimeter, the radius of the circumscribing circle, and the radius of a circle touching the base and the continuation of the other two sides, to construct the triangle.

Prob. XXIX. Given the vertical angle, and the base, to find the triangle.

the radii of two circles, each touching a side of the triangle and the continuation of the other side and the base, to describe it.

Prob. XXX. Given the vertical angle, the distance between the centres of the inscribed and circumscribing circles, and the rectangle under the base and radius of the inscribed circle, to construct the triangle.

Prob. XXXI. Given the vertical angle, the perimeter, and the solid whose base is the rectangle of the sides, and altitude the base of the triangle equal to a given cube, to determine the triangle.

Prob. XXXII. Given the vertical angle, and the perimeter of a plane triangle, to construct it, when the pyramid, whose base is the triangle, and altitude the radius of the circumscribing circle, is equal to a given cube.

Prob. XXXIII. Given the difference of the sides, and the radii of the inscribed circle and a circle touching the base and the continuation of the other two sides, to construct the triangle.

Prob. XXXIV. Given the perimeter of a triangle to construct it, when the solid contained under the radii of three circles, each touching a side of the triangle and the continuation of the other two, is a maximum.

Prob. XXXV. Given the radius of the inscribed circle, and the radii of two circles, each touching a side of the triangle and the continuation of the other two sides, to construct the triangle.

Prob. XXXVI. Given the ratio of the lines from the angles to the centre of the inscribed circle, and the perimeter, to construct the triangle.

Prob. XXXVII. Given the radii of three circles,

Three Propositions from *Lawson*.

(To be answered in Number VIII.)

IN AB, the diameter of a circle produced, let be taken the point C, and CD be perpendicular to AB, and therein be taken two points E and F, on different sides of C, such that the rectangle ECF may be equal to the rectangle ACB, and from the points E and F let EG, FG, be inflected to any point G in the circle, meeting the same in H and K, and let HK, when drawn, meet the diameter AB in L; then I say that

$$AL : LB :: AC : CB.$$

Let AB touch a circle in B, and any line AE be drawn equal to AB, and likewise from A let any line be drawn to cut the circle in C and D, and let EC, ED, be drawn meeting the circle again in F and G; then FG being drawn, will be parallel to AE.

Let AB touch a circle in B, and therein be taken two points E and F on the same side of A, B; such

such that the rectangle EAF may be equal to the square of AB, and from A let any line be drawn meeting the circle in C and D, and EC, FD, be drawn, meeting the circle again in G and H; then GH being drawn, will be parallel to AB.

ARTICLE IV.

Two Propositions from *Stewart's Theorems*.

(To be answered in Number VIII.)

PROP. XXV. THEO. XXII.

LET there be any figure given by position of a greater number of sides than three, and let a , b , c , &c. be given magnitudes, as many in number as there are sides in the figure; four right lines may be found that will be given by position, such, that if from any point there be drawn perpendiculars to the sides of the figure, and likewise there be drawn perpendiculars to the four lines found, the cube of the perpendicular drawn to one of the sides of the figure, together with the solid to which the cube of the perpendicular drawn to another of the sides of the figure has the same ratio that a has to b , together with the solid to which the cube of the perpendicular drawn to another of the sides of the figure has the same ratio that a has to c , and so on, will be equal to the solid to which the sum of the cubes of the perpendiculars drawn to the four lines found has the same ratio that four times a has to the sum of a , b , c , &c.

PROP.

PROP. XXVI. THEO. XXIII.

Let there be any regular figure inscribed in a circle, and from all the angles of the figure let there be drawn right lines to any point in the circumference of the circle; the sum of the fourth powers of the chords will be equal to six times the multiple of the fourth power of the semi-diameter of the circle by the number of the sides of the figure.

ARTICLE V.

On the Resolution of Indeterminate Problems.

By John Leslie, A. M.

(Continued from p. 367, vol. 1.)

PROBLEM VII.

LET c and d be known values of x and y in the expression, $ax^2 + b = y^2$, and from these let it be required to discover others.

Since $ax^2 + b = y^2$, and $ac^2 + b = d^2$, subtracting these equations, we shall obtain $ax^2 - ac^2 = y^2 - d^2$, and by resolution, $(ax - ac)(x + c) = (y + d)(y - d)$; whence $ax - ac = my - md$, and $x + c = (y + d) \div m$. From the first of these equations, $x = (my - md + ac) \div a$, and from the second, $x = (y + d - mc) \div m$; whence $(my - md + ac) \div a = (y + d - mc) \div m$, and $y = (m^2d + ad - 2mac) \div (m^2 - a)$, or $((m^2 + a)d - 2mac) \div (m^2 - a)$. But $x = (y + d) \div m - c$; therefore x

$\equiv (2md - (m^2 + a)c) \div (m^2 - a)$. To simplify these formulæ, put $p \equiv (m^2 + a) \div (m^2 - a)$, and $q \equiv 2m \div (m^2 - a)$; then will $x \equiv dq - cp$, and $y \equiv pd - qac$. If c become negative, the conditions of the problem will not be affected. In this case, $x \equiv pc + qd$, and $y \equiv pd + aqc$. The values of x and y obtained from either of these formulæ, may be repeatedly substituted for those of c and d ; and thus a variety of numbers will be discovered.

Suppose $2x^2 + 7 = y^2$, then $c \equiv 1$, $d \equiv 3$; and if $m \equiv 2$, $p \equiv (4 + 2) \div (4 - 2) \equiv 3$, and $q \equiv 4 \div (4 - 2) \equiv 2$; whence $x \equiv 3 \cdot 1 + 2 \cdot 3 \equiv 9$ or 3 , and $y \equiv 3 \cdot 3 + 4 \equiv 13$ or 5 . Again, $x \equiv 3 \cdot 9 + 2 \cdot 13 \equiv 53$ or 1 , and $y \equiv 3 \cdot 13 + 4 \cdot 9 \equiv 75$ or 3 . Or $x \equiv 3 \cdot 3 + 2 \cdot 5 \equiv 19$ or 1 , and $y \equiv 3 \cdot 5 + 4 \cdot 3 \equiv 27$ or 3 ; and so repeatedly.

We may observe, that the value of p is the same with that of y in Prob. VII. Case II. Cor. 2, and the value of q the same with that of x . Whence, if $p \equiv d$, and $q \equiv c$, we shall obtain for the expression $ax^2 - 1 = y^2$, $x \equiv 2cd$, and $y \equiv d^2 + ac^2$. Thus, in the example, $2x^2 - 1 = y^2$, where $c \equiv 2$, and $d \equiv 3$, $x \equiv 2 \cdot 2 \cdot 3 \equiv 12$, and $y \equiv 3 \cdot 3 + 2 \cdot 2 \cdot 2 \equiv 17$; and again, $x \equiv 2 \cdot 12 \cdot 17 \equiv 408$, and $y \equiv 17 \cdot 17 + 2 \cdot 12 \cdot 12 \equiv 577$.

PROBLEM IX.

To find two rational numbers, the sum of which shall be equal to a given number, and the sum of their squares a square.

By hypothesis, $x + y = a$, and $x^2 + y^2 = z^2$. Transposing the second equation, $x^2 = z^2 - y^2$, and resolving into factors, $x \times x = (z + y)(z - y)$; whence, $x \equiv mz - my$, and $z + y \equiv mx$; wherefore $mz \equiv my + x$, and $z \equiv (my + x) \div m$; also, $z \equiv mx - y$; consequently, $my + x \equiv m^2x - my$, and $y \equiv (m^2x - x) \div 2m$.

But

But from the first equation, $y=a-x$; wherefore $a-x=(m^2x-x)\div 2m$, and $x=2am\div(m^2+2m-1)$; consequently, $y=a(m^2-1)\div(m^2+2m-1)$, and $z=a(m^2+1)\div(m^2+2m-1)$.

Suppose $a=23$, and $m=4$; then $x=8$, $y=15$, and $z=17$. For $8+15=23$, and $64+225=289=(17)^2$.

PROBLEM X.

To find two numbers, whose sum shall be a given number and the product of the sums, formed by adding given numbers to them, a square.

By hypothesis, $x+y=a$, and $(x+b)(y+c)=z^2$. From the second equation, we obtain by assumption, $x+b=mz$, and $z=my+mc$; therefore, $z=(x+b)\div m=my+mc$; and so $x=m^2y+m^2c-b$. But from the first equation, $x=a-y$; consequently, $m^2y+m^2c-b=a-y$, and $y=(a+b-m^2c)\div(m^2+1)$; also $x=a-y=(am^2+m^2c-b)\div(m^2+1)$, and $z=(x+b)\div m=(am+mc+mb)\div(m^2+1)$.

Suppose $a=17$, $b=6$, $c=2$, and let $m=2$; then $y=(17+6-8)\div(4+1)=3$, $x=(68+8-6)\div(4+1)=14$, and $z=(34+4+12)\div(4+1)=10$. But $14+3=17$, and $(14+6)(3+2)=100=(10)^2$.

PROBLEM XI.

Let it be required to find two numbers, such that, if to each, their sum and difference, unit be added, the numbers resulting shall be squares.

The first condition will be observed, if the numbers be denoted by x^2-1 and y^2-1 . The hypothesis will then require $x^2+y^2-1=z^2$ and $x^2-y^2+1=v^2$.

Transposing the first equation, $x^2-1=z^2-y^2$,
and

and resolving into factors, $(x+1)(x-1)=(z+y)(z-y)$; whence, $x+1=mx-my$, and $z+y=mx-m$; therefore, $z=mx-m-y=(x+1+my)\div m$, from which we have $y=(m^2x-x-m^2-1)\div 2m$.

Again, transposing the second equation, $x^2-y^2=v^2-1$, and resolving, $(x+y)(x-y)=(v+1)(v-1)$ and by assumption, $x+y=pv-p$, and $v+1=px-py$, and therefore, $v=px-py-1=(x+y+p)\div p$. Hence $y=(p^2x-x-2p)\div(p^2+1)$.

But it was found, that $y=(m^2x-x-m^2-1)\div 2m$; wherefore $(p^2x-x-2p)\div(p^2+1)=(m^2x-x-m^2-1)\div 2m$; and by reduction, $x=(p^2m^2+p^2-4mp+m^2+1)\div(p^2m^2-2mp^2-p^2+m^2+2m-1)$, or $=((m^2+1)(p^2+1)-4mp)\div(p^2((m-1)^2-2)+(m+1)^2-2)$.

In the same manner, by finding the values of x in terms of y , &c. we obtain $y=(m^2p^2-2m^2p+p^2-m^2+2p-1)\div(p^2m^2-2mp^2-p^2+m^2+2m-1)$, or $=\{m^2((p-1)^2-2)+(p+1)^2-2\}\div p^2((m-1)^2-2)+(m+1)^2-2$.

PROBLEM XII.

To find three numbers, the product of any two of which, increased by unit, shall be a square.

By hypothesis, $xy+1=v^2$, $xz+1=s^2$, and $yz+1=w^2$.

1. Transposing the first equation, $xy=v^2-1$, and resolving $x\times y=(v+1)(v-1)$, whence $y=mv-m$, and $v+1=mx$; consequently, $v=(m+y)\div m=mx-1$, and $x=(2m+y)\div m^2$.

2. Again, transposing the second equation, $xz=s^2-1$, and resolving, $x\times z=(s+1)(s-1)$; whence, $z=ps-p$, and $s+1=px$; consequently, $s=(z+p)\div p=pz-1$, and reducing, $x=(z+2p)\div p^2$. But $x=$

$x = (2m + y) \div m^2$; wherefore $m^2z + 2m^2p = 2mp^2 + p^2y$, and $y = (m^2z + 2m^2p - 2mp^2) \div p^2$.

3. Moreover, by the third equation, $yz = w^2 - 1$; whence, $y \times z = (w + 1)(w - 1)$, and $y = qw - q$, and $w + 1 = qz$; wherefore, $w = (y + q) \div q = qz - 1$, and $y = q^2z - 2q$. But $y = (m^2z + 2m^2p - 2mp^2) \div p^2$; consequently $p^2q^2z - 2p^2q = m^2z + 2m^2p - 2mp^2$, and $z = (2p^2q + 2m^2p - 2mp^2) \div (p^2q^2 - m^2)$. Now, $y = q^2z - 2q$; whence by substitution, $y = (2m^2pq^2 - 2mp^2q^2 + 2m^2q) \div (p^2q^2 - m^2)$. And because $x = (2m + y) \div m^2$, we have also $x = (2pq^2 - 2m + 2q) \div (p^2q^2 - m^2)$.

Cor. Let $m = 1$, then the formulæ will be more simple; $x = (2pq^2 + 2q - 2) \div (p^2q^2 - 1)$, $y = (2pq^2 - 2p^2q^2 + 2q) \div (p^2q^2 - 1)$, and $z = (2p^2q + 2p - 2p^2) \div (p^2q^2 - 1)$.

There is a remarkable case in which the above formulæ do not directly apply, the numerators and denominators vanishing at the same time. It is when $m = 1$, $p = 2$, and $q = \frac{1}{2}$. For, by art. 3, $y = (2m^2pq^2 - 2mp^2q^2 + 2m^2q) \div (p^2q^2 - m^2) = (1 - 2 + 1) \div (1 - 1) = 0 \div 0$; wherefore the value of y may be expressed by any assumed number, n . But, by art. 1, $x = (2m + y) \div m^2 = y + 2$; whence, $x = n + 2$. Also, by art. 2, $x = (z + 2p) \div p^2 = (z + 4) \div 4$; therefore, $z + 4 = 4n + 8$, and $z = 4n + 4$. Thus, 2, 4, 12; for $2 \times 4 + 1 = 9$, $2 \times 12 + 1 = 25$, and $4 \times 12 + 1 = 49$.

PROBLEM XIII.

To find a cube which shall be equal to the product of a square by a given number.

By hypothesis, $x^3 = ay^2$, and resolving, $x \times x^2 = a \times y^2$; whence $x = ma$, and $y^2 = mx^2$; but $x^2 = (ma)^2$, consequently, $y^2 = m^3a^2$, and $y \times y = ma \times m^2a$; and by a second assumption, $y = pma$, and $m^2a = py$;

$=py$; but $x=ma$; whence $y=px$, and since $y=m^2a \div p$, $y=x^2 \div ap$; wherefore $x^2 \div ap=px$, and $x=ap^2$; but $y=px$, whence $y=ap^3$.

Suppose $a=3$, and $p=2$; then $x=3 \times (2)^2=12$, and $y=3 \times (2)^3=24$. For $(12)^3=1728=3 \cdot (24)^2$.

PROBLEM XIV.

To find two numbers, the sum of which shall be a given square, and the sum of their cubes a square.

By hypothesis, $x+y=a^2$, and $x^3+y^3=z^2$. Dividing the second equation by the first, we obtain $z^2 \div a^2 = x^2 - xy + y^2$, or $z^2 \div a^2 - y^2 = x^2 - xy$, and resolving into factors, $(z \div a + y)(z \div a - y) = x(x - y)$: whence, $x = m(z \div a - y)$, and $z \div a + y = m(x - y)$. By reducing the first of these expressions, $z = (ax + amy) \div m$; and by the second, $z = max - may - ay$; whence $(ax + amy) \div m = max - may - ay$, and $y = (m^2x - x) \div (m^2 + 2m)$. But from the first equation, $y = a^2 - x$; wherefore, $(m^2x - x) \div (m^2 + 2m) = a^2 - x$, and therefore, $x = a^2(m^2 + 2m) \div (2m^2 + 2m - 1)$. But $y = a^2 - x$, consequently, $y = a^2(m^2 - 1) \div (2m^2 + 2m - 1)$. Also, because $z = (ax + may) \div m$, we have by substitution, $z = a^2(m^2 + m + 1) \div (2m^2 + 2m - 1)$.

Cor. 1. If $a = 2m^2 + 2m - 1$, two whole numbers may be always found, the sum of which, and that of their cubes, shall be squares. For in this case $x = (2m^2 + 2m - 1) \div (m^2 + 2m)$, $y = (2m^2 + 2m - 1) \div (m^2 - 1)$, and $z = (2m^2 + 2m - 1)^2(m^2 + m + 1)$.

Thus, if $m=2$, we shall find $x=88$, $y=33$, and $z=847$. But $88+33=121=(11)^2$, and $(88)^3+(33)^3=717409=(847)^2$.

Cor. 2. If y be negative, we shall obtain two numbers,

numbers, the difference of which, and that of their cubes, shall be squares.

Put $m = p \div q$, and substituting

$$x = a^2 \left(\frac{p^2}{q^2} + \frac{2p}{q} \right) \div \left(\frac{2p^2}{q^2} + \frac{2p}{q} - 1 \right),$$

$$y = -a^2 \left(\frac{p^2}{q^2} - 1 \right) \div \left(\frac{2p^2}{q^2} + \frac{2p}{q} \right), \text{ and}$$

$$z = a^3 \left(\frac{p^2}{q^2} + \frac{p}{q} + 1 \right) \div \left(\frac{2p^2}{q^2} + \frac{2p}{q} - 1 \right), \text{ and}$$

by reduction, $x = a^2(p^2 + 2pq) \div (2p^2 + 2pq - q^2)$, $y = a^2(q^2 - p^2) \div (2p^2 + 2pq - q^2)$, and $z = a^3(p^2 + pq + q^2) \div (2p^2 + 2pq - q^2)$. If $a = 2p^2 + 2pq - q^2$, we shall obtain whole numbers; for $x = (2p^2 + 2pq - q^2)(p^2 + 2pq)$, $y = (2p^2 + 2pq - q^2)(q^2 - p^2)$, and $z = (2p^2 + 2pq - q^2)(p^2 + pq + q^2)$.

These examples will probably be thought sufficient to explain the application of this method to the solution of indeterminate problems in general, and to shew that it is not less extensive, and much more uniform, than those that are commonly in use.

ARTICLE VI.

Useful Propositions in Geometry.

By Mr. M. A. HARRISON.

(Continued from page 369, vol. 1.)

PROP. IX. THEO. Fig. 175, Plate 12.

THE distance (TU) between the points where perpendiculars from the angles at the base meet

meets the line (CE) bisecting the vertical angle, is bisected by the perpendicular (l Q) drawn through (L) the middle of the base, upon the said bisecting line.

Demon. Since GT, AU, are perpendicular to CE, they are parallel to each other;

but PQ is perpendicular to CE;

hence, GT, PQ, AU, are parallel to each other;

consequently, $CG : CT :: GP : TQ :: PA : QU$;

but, $GP = PA$, therefore $TQ = QU$.

Q. E. D.

PROP. X. THEO.

If LD be bisected in n, and the perpendicular no demitted to meet PQ produced in o: then I say o will be the centre of a circle passing through the points L, T, D, U.

Demon. It has been demonstrated, that the points L, T, D, U, are in the periphery of a circle; and LD, TU, are chords.

But by hyp. $Ln = nD$, and no is \perp to LD,

therefore, no passes through the centre.

Again, by Prop. IX. $TQ = QU$, and Qo is \perp to TU;

therefore, Qo passes through the centre.

Q. E. D.

PROP. XI. THEO.

If LU be joined, it will be parallel to BC, and equal to half the difference of the sides.

Demon. By the last proposition, a circle described about o as a centre, with the distance oL, will pass through the points L, T, D, U;

Also by Prop. IX. $TQ = QU$, and QL is \perp to TU;

therefore the \angle s LUT, LTU, standing on = arches, are =;

but, by \parallel s, $\angle LTU = \angle BCT$;

hence,

hence, $\angle LUT = \angle BCT$;
therefore, LU, BC , are parallel.

And because the \perp of an isosceles Δ bisects the base, UL will be $= LT = AP (PG) =$ half the diff. of AC, CB .

Q. E. D.

PROP. XII.

If DT be joined, the \angle s DTB, DCB , will be equal to each other.

Demon. The \angle s $LDT, LUT, (LTU)$ standing on the same arch will evidently be equal. And
 $DBT + DTB = LDT = TCB = TCD + DCB$;
but, $TCD = DBT$ (Prop. II.);
therefore, $DTB = DCB$.

Q. E. D.

Cor. 1. The $\angle LDT = \frac{1}{2} \angle ACB$; also, $\angle TDC = \angle TBC$, and the points T, C, B, D , are in a circle.

Cor. 2. $LD \times TU = (TB + DU) \times TU (LU)$; (DU being joined).

Cor. 3. If IR be joined, it will be \perp to TD .

Cor. 4. If DU be drawn; DT, DU , will be parallel to AE, BE , respectively.

Cor. 5. The Δ s ACB, UDT , are equiangular; as also are the Δ s ALU, CTD ; and DBT, UEL .

(To be Continued,)

ARTICLE VII.

Demonstration of Article XXXVIII. Vol. I.

By Mr. Thomas Keith, the Proposer.

LET the figure (202, pl. 13.) be constructed, as in page 283, vol. 1. Imagine IP and IZ to be joined by elliptical arches; then will the angle IZP , or its supplemental angle IZp , be the azimuth; we have, therefore, only the angle IZp to measure.

With O as a centre, and radius Oa , describe the arch ara , and produce IR to r . Join Or , then the plane angle ROI is equal to the spherical angle IZp (*Cratell's Translation of Mauduit's Trigonometry, page 112.*)

Now, $Cm = Or = Oa$, by construction, and $\angle C = \angle OI$, because RI is parallel to ON , and IO parallel to $\angle C$; therefore the triangles $\angle Cm$ and $\angle Or$ are equal to each other (*Simpson's Geometry, I. 17.*): hence the angle $\angle Cm$, measured by the arch Sm , is equal to the angle $\angle Or$, which measures the spherical angle IZp .

Q. E. D.

The same by Mr. Lowry.

This method of constructing an azimuth is an evident consequence of the orthographic projection of the sphere on the plane of the meridian. For, let Z be the zenith, P the pole, A the object, aAa the parallel of altitude, pAp the parallel of declination, SDL the horizon, EQF the equinoctial, and let the rest of the lines and arches be drawn as in the figure. Then ZP is the colatitude, ZA , or Za , the coaltitude, and PA ,

PA, or Pp, the co-declination; and by the nature of the orthographic projection, the intersection I of the diameters aa , pp , of the lesser circles aAa , pAp , will be the projection of the point A, and the ellipsis described through the points Z, I, and meeting SCL in B, will be the representation of the arch ZAD upon the plane ZNRS.

Join Bn, then (*Walker on the Sphere*, B. IV. Prop. IV. Cor. 1.) $IO(Cx) : Oa :: BC : SC$; but, by *conf.* $Cm = Oa$, and $Cn = CS$; therefore, $Cx : Cm :: BC : CN$; wherefore nB is parallel to xR .

Now, since Z is the pole of LDS, the arch DL is the measure of the angle DZL, and, because L is the pole of the great circle ZCN, and Bn is parallel to CN, and B is the projection of the point D, it follows, that nL is equal to DL; therefore, the arch nL is the measure of the angle AZP, or the azimuth from the North, and, consequently, the arch nS (the supplement of nL) is the azimuth from the South. Whence, Mr. Keith's construction is evident.

Q. E. D.

ARTICLE VIII.

Demonstrations of Mr. EMERSON'S FORMS of FLUENTS, by the Rev. L. Evans, Foxfield.

THE finding a fluent from a given fluxion, is certainly the most difficult and most useful problem in the whole doctrine of fluxions. There are direct and general rules for finding the fluxion of any quantity, be it ever so compounded;

the area SDBV ; therefore ϕ or the fluent of $x^{-1} \dot{x}$, is $= \text{area} \div R^2$.

FORM III. demonstrated.

This is evident, for $(\alpha + \beta z^n)^{\mu+1} \div (\mu+1)\beta n$ put into fluxions, is $= (\alpha + \beta z^n)^{\mu} z^{n-1} \dot{z}$ the given fluxion. It may be proved otherwise, thus :—Put $\alpha + \beta z^n = x$, then $z^n = (x - \alpha) \div \beta$; this in fluxions gives $z^{n-1} \dot{z} = \dot{x} \div n\beta$; now substituting x , and $\dot{x} \div n\beta$ for their equals in the given fluxion, and it becomes $x^{\mu} \dot{x} \div n\beta$, and the fluent found by form 1, is $x^{\mu+1} \div (\mu+1)n\beta$, hence, by restoring the value of x , ϕ will be $= (\alpha + \beta z^n)^{\mu+1} \div (\mu+1)n\beta$. When $\mu = -1$, the fluxion becomes $x^{-1} \dot{x} \div n\beta$, whose fluent cannot be found by this form, as is evident from form 1.

FORM IV. demonstrated.

1st. To prove $\phi = (L \div n\beta) \times \log. \text{ of } (\alpha + \beta z^n)$. Put $\alpha + \beta z^n = x$, then $\dot{x} = n\beta z^{n-1} \dot{z}$, and $x^{-1} \dot{x} = \dot{x} \div x = n\beta z^{n-1} \dot{z} \div (\alpha + \beta z^n)$, therefore, by form 2, the fluent of $n\beta z^{n-1} \dot{z} \div (\alpha + \beta z^n) = L \times \log. \text{ of } (\alpha + \beta z^n)$; and therefore,

therefore ϕ , or the fluent of $z^{n-1}z \div (a + \beta z^n) = (1 \div \beta) \times \log. of (a + \beta z^n)$;

2d. To prove $\phi = \text{area} \div n\beta R^2$ of an hyperbola between the asymptotes, whose inscribed parallelogram is R^2 , abscissa $a \div \beta + z^n$.

Assume $a \div \beta + z^n = x$, then $x = nz^{n-1}z$, and $x^{-1} = 1 \div nz^{n-1} \div (a \div \beta + z^n)$; therefore, by, form 2, the fluent of $nz^{n-1}z \div (a \div \beta + z^n)$, or its equal, $\beta nz^{n-1}z \div (a + \beta z^n)$, is $= \text{area} \div R^2$ of an hyperbola, &c. therefore ϕ , or the fluent of $z^{n-1}z \div (a + \beta z^n)$ is $= \text{area} \div n\beta R^2$ of an hyperbola between the asymptotes, &c.

FORM V. demonstrated.

1st. To prove $\phi = (2N \div n\sqrt{a\beta}) \times \text{degrees in that arch of a circle whose rad. is 1 and nat. tangent } \sqrt{\beta z^n} \div a$.

If r be the rad. and t the tangent of an arch, the fluxion of that arch will be $r^2t \div (r^2 + t^2)$; and when $r^2 = a$, and $t^2 = \beta z^n$, t will be $= \frac{1}{2}n\beta^{\frac{1}{2}} z^{\frac{n-1}{2}}z$; therefore, by substitution, the fluxion of the arch becomes $\frac{1}{2}na\beta^{\frac{1}{2}} z^{\frac{n-1}{2}}z \div (a + \beta z^n)$; the fluent of which is evidently the arch whose rad. is \sqrt{a} , and nat.

nat. tan. $\sqrt{\beta z^n}$; therefore ϕ , or the fluent of $z^{\frac{1}{2}n-1} z \div (\alpha + \beta z^n)$ is $= (2 \div n \alpha \sqrt{\beta}) \times \text{arch}^*$ whose rad. is $\sqrt{\alpha}$, and nat. tan. $\sqrt{\beta z^n} = (2N \div n \sqrt{\alpha \beta}) \times \text{degrees in the arch whose radius is } \sqrt{\alpha}$, and tan. $\sqrt{\beta z^n} = (2N \div n \sqrt{\alpha \beta}) \times \text{degrees in the arch whose rad. is } 1$, and tan. $\sqrt{\beta z^n} \div \alpha$.

2d. To prove $\phi = \text{arch} \div \frac{1}{2} n R \sqrt{\alpha \beta}$ of a circle whose rad. is any line R , and tangent $R \sqrt{\beta z^n} \div \alpha$. It has been shewn that $\phi = (2N \div n \sqrt{\alpha \beta}) \times \text{deg. in the arch whose rad. is } 1$, and tangent $\sqrt{\beta z^n} \div \alpha$, and therefore $= ((2NR \div n R \sqrt{\alpha \beta}) \times \text{degrees, or}) (\text{arch} \div \frac{1}{2} n R \sqrt{\alpha \beta})$ of a circle whose rad. is 1 , and tan. $R \sqrt{\beta z^n} \div \alpha$.

3d. To prove $\phi = (4 \text{ sectors} \div n R^2 \sqrt{\alpha \beta})$ of the circle whose rad. is R , and tan. $R \sqrt{\beta z^n} \div \alpha$.

Now, $\text{arch} \times R = 2 \text{ sectors}$, or $\text{arch} = (4 \text{ sectors} \div 2R)$, this substituted for the arch in the last case, gives $\phi = (4 \text{ sectors} \div n R^2 \sqrt{\alpha \beta})$ of the circle whose rad. is R , and tan. $R \sqrt{\beta z^n} \div \alpha$.

FORM VI. *demonstrated.*

1st. To prove $\phi = (L \div n \sqrt{\alpha \beta}) \times \log. \text{ of } (\sqrt{\alpha} + \sqrt{\beta z^n}) \div (\sqrt{\alpha} - \sqrt{\beta z^n})$. Put $\alpha = b^2$, and $\beta z^n = x^2$.

* The length of any arch of a circle whose rad. is R , is $= N \times R \times \text{degrees in that arch.}$

† For, $\sqrt{\alpha} : \sqrt{\beta z^n} :: 1 : \sqrt{\beta z^n} \div \alpha$.

‡ For, $1 : \sqrt{\beta z^n} \div \alpha :: R : R \sqrt{\beta z^n} \div \alpha$.

then

then $z^{\frac{1}{2}n} = x \div \beta^{\frac{1}{2}}$, and $\frac{1}{2}nz^{\frac{1}{2}n-1}z = x \div \beta^{\frac{1}{2}}$, there-

fore ϕ becomes $2x \div n\beta^{\frac{1}{2}}(b^2 - x^2) = 2bx \div b\eta\beta^{\frac{1}{2}}$

$(b^2 - x^2) = (1 \div n\beta^{\frac{1}{2}}) \times ((x \div b+x) - (-x \div b-x))$;

Now the fluent of $x \div (b+x)$ is $= L \times \log. (b+x)$;

and the fluent of $-x \div (b-x)$ is $= L \times \log. (b-x)$;

therefore $\phi = (L \div n\beta^{\frac{1}{2}}) \times (\log. \text{ of } b+x - \log. \text{ of } b-x)$

$= (L \div n\beta^{\frac{1}{2}}) \times \log. \text{ of } (b+x \div b-x) = (\text{by}$

restoring the values of b and x) $(L \div n\sqrt{a\beta}) \times \log. \text{ of } (\sqrt{a} + \sqrt{\beta z^n}) \div (\sqrt{a} - \sqrt{\beta z^n})$.

2d. ϕ is evidently $= (L \div n\sqrt{a\beta}) \times \log. \text{ of } (\sqrt{\beta z^n} + \sqrt{a}) \div (\sqrt{\beta z^n} - \sqrt{a})$, by changing the signs of the quantities of the preceding case.

3d. ϕ is proved $= (L \div n\sqrt{a\beta}) \times \log. \text{ of } (x + \beta z^n + 2\sqrt{a\beta z^n}) \div (x - \beta z^n)$, by multiplying the numerator and denominator in the 1st case by $\sqrt{a} + \sqrt{\beta z^n}$.

4th. In the same way this case is proved by multiplying the numerator and denominator in the second case by $\sqrt{a} - \sqrt{\beta z^n}$.

5th. ϕ is evidently $= (\text{area} \div nR^2\sqrt{a\beta})$ of the hyperbola between the asymptotes whose inscribed parallelogram is R^2 , and abscissas terminating this area $\sqrt{a} + \sqrt{\beta z^n}$ and $\sqrt{a} \vee \sqrt{\beta z^n}$.

6th. To prove $\phi = \{4 \text{ sectors} \div nR^2\sqrt{a\beta}\}$ of a right-angled hyperbola whose semi-transverse is R , and tangent at the vertex $R\sqrt{\beta z^n} \div a$. Put

Put a for the semi-transverse and semi-conjugate, and t for the tangent at the vertex: then by the property of the hyperbola $\frac{1}{2} a^2 \times t \div (a^2 - t^2)$ is = the fluxion of the sector. Now substitute R and $R \sqrt{\beta z^n} \div a$ for a and t , then t becomes $\frac{1}{2} n R \sqrt{\beta \div a} \times z^{\frac{1}{2}n-1} z$, and the fluxion of the sector = $\frac{1}{2} R^2 \times (\frac{1}{2} n R \sqrt{\beta \div a} \times z^{\frac{1}{2}n-1} z) \div (R^2 - R^2 \times \beta z^n \div a)$ = $(n R^2 \sqrt{\beta \div a} \times z^{\frac{1}{2}n-1} z) \div 4(a - \beta z^n)$; therefore the fluent of $z^{\frac{1}{2}n-1} z \div (a - \beta z^n)$ = $(4 \text{ sectors} \div n R^2 \sqrt{a\beta})$ of a right-angled hyperbola, &c.

ARTICLE IX.

Answers to the Mathematical Questions proposed in
ARTICLE XLIII. No. IV. Vol. I.

IN QUESTION 69, answered by Tamerlane.

LET EABCD in fig. 203, pl. 15, represent the bastion, in which $BC = BA$ (the faces of the bastion) is given = 60 yards, and the angles (of the shoulders) EAB, BCD, and the salient angle ABC each = 110° , to find (the flanks) $CD = AD$ when (the gorge) $ED = 80$ yards.

Here the $\angle BCA = BAC = \frac{1}{2}(180^\circ - 110^\circ) = 35^\circ$; hence, $s.BAC : BC :: s.ABC : AC = 98.29825$ yds. Then the $\angle dCD = 110^\circ - 35^\circ = 75^\circ$, $CDd = 90^\circ - 75^\circ = 15^\circ$, and $DC = \frac{1}{2}(AC - ED) = 9.149125$; therefore $s.dDC : dC :: s.DdC$ (rad. $\therefore DE = AE = 35.3495$ yards the length of each flank as required.

Other ingenious answers were received from Messrs.
Gregory, Simpson, Swale and Thornoby.

II. QUESTION 70, answered by Mr. R. Simpson,
Croxdale, Durham.

In fig. 47, pl. 3, take $DC = 30$ feet, (not 50 as printed) and make DE perpendicular thereto and $= 5\frac{1}{2}$ feet; join EC and draw DB, EB making the \angle s $CDB, CEB = 40^\circ 30'$ and $50^\circ 40'$ respectively. From B , the intersection of DB, EB , demit, on DC produced, the $\perp BA$; then AC will be the breadth of the river and AB the height of the spire.

Calculation. In the right $\triangle CDE$ there is given CD, DE , to find the $\angle DEC$, which added to the $\angle BEC$ gives the $\angle DEB$. Again the $\angle CDB$ taken from a right \angle leaves the $\angle BDE$. Then in the $\triangle DEB$ there is given all the \angle s and the side DE , to find DB . Whence in the right $\triangle ABD$ we have given the side BD and the $\angle ADB$ to find $AB = 702.622$ feet the height of the spire; and $AC (= DA - DC) = 792.6647$ feet the breadth of the river.

This Question was also ingeniously answered by Messrs. Lee, Lowry, Swale, Tamerlane and Thornoby.

III. QUESTION 71, answered by Mr. Simpson.

Let half the given area be represented by a , the absciss by x , and the ordinate by y . Then by page 363 (2d. Ed.) of Dr. Hutton's Mensuration, $\sqrt{(\frac{4}{3}x^2 + y^2)}$ = the length of the single curve; therefore by the question $\sqrt{(\frac{4}{3}x^2 + y^2)} + y$ (= half the perimeter) must be a minimum, and its fluxion $= 0$, that is, $(\frac{4}{3}xx + yy) \div (\sqrt{(\frac{4}{3}x^2 + y^2)}) + y = 0$. But every parabola is $= \frac{2}{3}$ of its circumscribing parallelogram, therefore $x = 3a \div 2y, x^2 = 9a^2 \div 4y^2$ and $\dot{x} = -3ay \div 2y^2$; hence, substituting for x, x^2 and \dot{x} their respective equals, and dividing by y , we get $-3a^2 \div y^3 \sqrt{(3a^2 \div y^2 + y^2)} + y \div \sqrt{(3a^2 \div y^2 + y^2)} + 1 = 0$. This equation, by proper reduction becomes $y = \sqrt{a} = 49.1934$ yards. Hence $x = 73.7802$;

7802; the length of the single curve ≈ 98.3769 ; consequently, the whole perimeter (\equiv the double curve $+$ the double ordinate) ≈ 295.1406 yards, which at 4d. per yard, comes to 4l. 18s. $4\frac{1}{2}$ d. And from the above dimensions the parabolic field may be easily described.

Other answers were also given by Messrs. Lowry, Swale, Tamerlane and Thornoby.

IV. QUESTION 72, answered by Mr. Johnston, the Proposer.

Let x, y and z be the numbers required; x being the greatest.

Assume 1. $x - y = m^2$,

2. $x - z = n^2$,

then 3. $y - z = n^2 - m^2$.

By harmonic proportion we have $x : z :: x - y : y - z$; or, $xz = xy - xz + zy$.

Again, by multiplying the first and third equations together, $xy - xz + yz = m^2n^2 - m^4 + y^2$; therefore $xz = m^2n^2 - m^4 + y^2$; add four times this equation to the square of the second, and we have $x^2 + z^2 = 4m^2n^2 - 4m^4 + n^4 + 4y^2$. Hence $2x = \sqrt{(4m^2n^2 - 4m^4 + n^4 + 4y^2)} + n^2$; but $2x = 2m^2 + 2y$; therefore $2m^2 + 2y = \sqrt{(4m^2n^2 - 4m^4 + n^4 + 4y^2)} + n^2$. Whence $y = 2m^2(n^2 - m^2) \div (2m^2 - n^2)$, where $n^2 - m^2$ must be a square number, and $2m^2$ greater than n^2 .

Suppose $n = 5$, and $m = 4$, then $x = 400 \div 7$, $y = 288 \div 7$ and $z = 225 \div 7$, or multiplying by 7, the numbers become 2800, 2016, and 1575.

Messrs. Gregory and Lowry solved it in very nearly the same manner, and brought out the same result.

V. QUESTION 73, answered by Mr. M. A. Harrison.

It is well known that the quantity of any fluid overflowing the containing vessel when full, in consequence of the immersion of any solid, is universally equal

equal to the number of cubic inches in the part or whole of the solid so immersed; and consequently in the present case, the immersed part of the cylinder must be a maximum. Now this being admitted, let ACDBEF (fig. 204, pl. 15.) represent a section of the sphere, of which the given vessel is one-half; draw the diameter AB representing the water's surface, and parallel thereto apply the chord CD equal the given length of the cylinder; perpendicular to CD draw DE meeting AB in G and the periphery in E; so shall DE be the required diameter of the cylinder.

For, the cylinder whose length is CD and diameter DE, is evidently the greatest that can be inscribed in the same sphere, and of the same altitude CD: but $DG = GE$; hence the part immersed is equal to half the whole cylinder, and consequently the greatest possible; for it is well known that *halves* are as their *wholes*.

Calculation. By the circle $AG \cdot GB = GD^2 (= GE^2)$; then $DE = 2\sqrt{AG \cdot GB} = 2\sqrt{40 \times 8} = 16\sqrt{5} =$ the diameter required.

The same by Mr. Johnston, Birmingham.

Since the length of the leaden cylinder is given, it is obvious that the part immersed will be a maximum (or the greater quantity of water will overflow) when the area of the end of that part of the cylinder immersed in the water is the greatest possible. This will evidently happen when half the cylinder is immersed in the fluid, for then the edges of the ends of the cylinder touch the vessel quite round. Now $AB = 48$, $CD = 32$; hence $AI = 8$ and $IB = 40$; therefore $CI = \sqrt{AI \cdot IB} = 8\sqrt{5} =$ the radius of the cylinder's base.

True solutions were also received from Messrs. Evans, Gregory, Lee, Lowry, Simpson and Swale.

To VI. QUESTION 74, we have received no satisfactory answer. If any of our correspondents will favour us with a solution it shall be inserted in a future Number. The Proposer's solution is also requested.

VII. QUESTION 75, answered by Mr. William Peacock, Land Surveyor, at Birmingham.

Put $g = 16\frac{1}{2}$ feet, then $2\sqrt{ga} =$ the velocity acquired by the ball in descending the height a , and were the ball perfectly elastic, it would ascend again to the same height. Let its ratio to perfect elasticity be as $1 : m$, then it is evident that the velocity on reflection will be diminished in the ratio of $1 : m$, and will be equal $2\sqrt{ga} \div m$; with this velocity the ball will ascend to a certain height, and descend again with the same velocity. This velocity will be again diminished in the same ratio of $1 : m$, and will be $= 2\sqrt{ga} \div m^2$. In the same way it appears that the velocity at the third reflection will be $= 2\sqrt{ga} \div m^3$. Now, the distance is $=$ the square of the velocity divided by $4g$; therefore the distance is $= a \div m^6$; but, this is per question $= \frac{2}{3}a$, therefore $m^6 = \frac{3}{2}$, and $m = \sqrt[6]{\frac{3}{2}} = 1.07$, &c.

Otherwise by Mr. James Lee, London.

Instead of the ball falling upon the plane, we will suppose it to impinge directly upon an equal and perfectly non-elastic ball, at the given distance a : since the result will not, by such a supposition, be any way affected. Now put $16\frac{1}{2} = c$, and the ratio to be determined as x to 1 . Then, the velocity of the first impact is evidently $= 2\sqrt{ac}$; and from the laws of collision, the velocity of the percutient body after the third stroke or impact is $= (1 + 2x)(1 - x)^2 \sqrt{ac} \div 8(1 + x)$.

But this velocity is manifestly the same as would be acquired by a heavy body in descending a space $= \frac{2}{3}a$; therefore $(1 + 2x)(1 - x)^2 \sqrt{ac} = 16(1 + x)\sqrt{\frac{2}{3}ac}$; from which equation we obtain $x = .93$, and consequently the required ratio is that of 93 to 100.

Ingenious solutions to this Question were also given by Messrs. Lowry, Simpson, Swale and Thornoby.

.... **VIII.**

VIII. QUESTION 76, answered by Mr. O. G. Gregory, Teacher of the Mathematics, Cambridge.

Either the proposer of this question has been inattentive to the manner of wording it; or, through haste in transcribing he has omitted something; or there is a press error; for as it now stands, it will admit of various answers, as persons may take up various suppositions. The solid which is to be cemented to the frustum when suspended, may be fixed either above it, below it, or at the less end, so that the axes of the two solids may form one right line;—perhaps the latter may be the meaning of the proposer. Yet, even then, some will imagine that the part to be added is not a similar solid, but the part necessary to complete the cone: whilst others may suppose the part added must be a similar frustum of a cone. I will give a solution on each supposition.

And first, let it be supposed that the part to be added is to complete the cone. In fig. 200, pl. 15, let VBC represent a section of the whole cone from the vertex to the diameter of its base; and ABCD the frustum given, in which $AD = \frac{1}{2}BC$. It is evident, that when the cone is completed $VK = KL$: therefore as C the point in the axis to be suspended by the chain is in the midway between K and L, $VC = \frac{1}{4}VL$. But it is known that the centre of gravity of a homogeneous cone is at the distance of $\frac{1}{4}$ ths of its axis from the vertex, and of course, VAD, the part to be cemented, must be of the same specific gravity as the given frustum.

In the second place, let it be imagined that the part to be added is a similar frustum; here a difficulty arises, as none of its dimensions are given. I will suppose that its greater base is equal to the less base of the given frustum: then will $AD = 2$, $EF = 1$ and $MK = 4$. Now the question amounts to this: given the dimensions of three conic frustums $PBIC$, $D-2$ $AD-1$

APDI, and EAFD, cemented together as per figure, and the specific gravities of the two larger equal to each other, to find the specific gravity of the smaller, so that the whole compound body EBCF shall remain in equilibrio when that part of the axis is suspended which is the centre of the circle that divides the two larger frustums.

Suppose f , g and G to be the centres of gravity of the three frustums, and let the weight of EADF be denoted by F , that of APID by w , and that of PBCI by W . Then, because the weight of every body may be considered as acting in its centre of gravity, we shall, from the nature of the common centre of gravity of a system of bodies, have $CG \times W = Cf \times F + Cg \times w$. The solidity of the three frustums are, by the rules of mensuration, found 7.3304, 19.8968, and 38.7464 inches respectively: and by means of the rule for determining the centre of gravity of bodies, we find $fC = 5\frac{1}{4}$, $gC = 1\frac{1}{4}$, and $GC = 2\frac{1}{4}$. If the specific gravities of the two larger frustums be denoted by 1, their weights will be expressed by their solidities. And as the weight F of the smaller frustum, is equal to the product of its solidity S into its specific gravity D , we shall, by substituting SD for F in the above equation, after proper reduction, get $D = (W \times GC - w \times gC) \div S \times fC = (38.7464 \times 2\frac{1}{4} - 19.8968 \times 1\frac{1}{4}) \div (7.3304 \times 5\frac{1}{4}) = 1.23076$ the specific gravity of the part to be added, that of oak being 1. But when oak is compared with water as a standard, its specific gravity is 925, water being 1000, and then the specific gravity of the piece to be cemented on is 1138, being nearly the same as a mixture of equal portions of rosin and pitch.

N. B. The rule which I make use of to find the centre of gravity of a conic frustum, is not a new one; but, as I believe it is not generally known, I think it may deserve a place in the Repository.

Put

Put D = greater diameter of the frustum, d = less diameter, h = half difference of the diameters, and L = length of the frustum: then $L \times (6h^2 + 8hd + 3d^2) \div (8h^2 + 12hd + 6d^2)$ = distance of the centre of gravity from the less base, measuring on the axis.

Other ingenious solutions were also received from Messrs. Harrison, Lee, Lowry and Swale.

IX. QUESTION 77, answered by Mr. J. Harris;
*Land-Surveyor and Teacher of the Mathematics,
at Caermarthen.*

Put x for the nat. cosine of the declination, then will $\sqrt{1-x^2}$ = its sine; $\sqrt{\frac{1}{2}(1-x)}$ = sine of the alt. at 6. Whence, by spherics $\sqrt{1-x^2} : 1 :: \sqrt{\frac{1}{2}(1-x)} : \sqrt{1 \div (2+2x)}$ = sine of latitude; and $\sqrt{1 - 1 \div (2+2x)}$ = its cosine. Now, the sine of the sun's meridian alt. = sine of (the declination + the complement of the latitude) = cos. dec. \times cos. lat. + sine dec. \times sine latitude; and that of the midnight depression = sine of (the complement of the latitude - the dec.) = cos. dec. \times cos. lat. - sine dec. \times sine lat. and their difference is = $2 \times$ sine dec. \times sine lat. that is = $2\sqrt{1-x^2} \times \sqrt{1 \div (2+2x)} = \sqrt{2-2x}$, which by the question must be = sine of half the latitude; hence this equation, $\sqrt{2-2x} = \sqrt{\frac{1}{2}(1 - \sqrt{1 - 1 \div (2+2x)})}$, which being reduced gives $x^3 - \frac{1}{2}x^2 - x + \frac{1}{2} = 0$. Solved $x = .9658791$ = nat. cosine of $15^\circ 0' 37''$ = sun's declination; and the latitude of the place is $30^\circ 17' 19''$.

The same by the Rev. Mr. L. Evans,

Let $x =$ fine alt. at 6, then will $\sqrt{1-x^2} =$ its cosine; $2x\sqrt{1-x^2} =$ fine of declination, and $1-2x^2 =$ its cosine. Now by spherics $2x\sqrt{1-x^2} : 1 :: x : 1 \div 2\sqrt{1-x^2} =$ fine of latitude : and $\sqrt{(3-4x^2) \div (4-4x^2)} =$ its cosine.

Then $(1-2x^2) \cdot \sqrt{\frac{3-4x^2}{4-4x^2} + \frac{2x\sqrt{1-x^2}}{2\sqrt{1-x^2}}} =$ fine of meri. alt.

and $(1-2x^2) \cdot \sqrt{\frac{3-4x^2}{4-4x^2} - \frac{2x\sqrt{1-x^2}}{2\sqrt{1-x^2}}} =$ fine of depreff.

The difference of these is $2x$, which by the question must be $=$ to the fine of half the latitude, that is $2x = \sqrt{\frac{1}{2}(1 - \sqrt{(3-4x^2) \div (4-4x^2)})}$. This equation expanded, &c. $x^6 - \frac{1}{2}x^4 + \frac{1}{4}x^2 - \frac{1}{2^{1/2}} = 0$. Hence $x = .130655$ the nat. fine of $7^\circ 30' 18'' .5$; $2x\sqrt{1-x^2} = 15^\circ 0' 37'' =$ sun's declination, the time answering to which is the 30th of April, or the 11th of August, and $1 \div 2\sqrt{1-x^2} = 30^\circ 17' 10'' .5$ the latitude.

This Question was also answered by Messrs. Lowry, Simpson and Swale.

X. QUESTION 78, answered by Mr. Swale, Leeds.

Let AGP (fig. 208, pl. 15,) represent the cylindrical monument inclined to the horizon BP, as by the question; take $PG = \frac{2}{3} PA$, and with the centre P and radius PG describe the quadrantal arch BGC: then from known principles, the time of
the

the cylinder's descent will be equal that in which a heavy body would describe the circular arch GB. To determine which, draw GM parallel to CP; put $5'' = a$; $3.1416 = 4n$, $16\frac{1}{2} = c$, cosine of $70^\circ = m$; and $PG = x$. Then the time in which a heavy body would gravitate thro' the diameter =

$\sqrt{2x \div c}$, and it has been determined that the times of describing the diameter and any arch GB whose

cosine is $= m$, to radius 1; is as $\frac{1}{n}$ to $1 + \frac{1-m}{8} +$

$\frac{(1-m)^2 \times 9}{64 \times 4} + \frac{(1-m)^3 \times 225}{6 \times 36 \times 8} + \&c.$ Hence, we shall

have $\frac{1}{n} : \frac{1-m}{8} + \frac{(1-m)^2 \times 9}{64 \times 4} + \&c. :: \sqrt{\frac{2x}{c}} : a$;

from which $x = a^2 c \div 2n^2 \times \left(1 + \frac{1-m}{8} + \frac{(1-m)^2 \times 9}{64 \times 4} + \&c. \right)^2$

$= 277.8$ feet, then $AP = \frac{3x}{2} = 416.7$ feet $= 138.9$

yards, the height of the monument.

This Question was also answered by Mr. Lowry.

XI. QUESTION 79, answered by Mr. Louis Hill.

The lines being drawn as directed in the question (fig. 209, pl. 15.) through the points A and B draw AP, BR, parallel to DE, meeting EF in P and R; draw CE, on which demit the \perp s LG, FH, and SQ, and let SK be produced (if necessary) to meet CE in T. Draw IEO \perp to DE, and CMN to CE, meeting DE in M, and EF in N.

Put $a = RB$, $c = AP$, $b = ER$, $m = EN$, $n = NC$, $r = EC$, $p = EM$, $q = MC$, $s = EF$, $x = CQ$, and $y =$

$y = SQ$. Then by sim. Δs , $LG = (q \div p) LE$. And since the points A, B, are equidistant from DE, OE will be $= IE$, and $PE = ER$; therefore by sim. Δs , $LE : EF :: BR : FR = FE + ER = z + b$, or $LE = za \div (z + b)$; therefore $LG = qza \div p(z + b)$, $EG = rza \div p(z + b)$, and $CG = EG - CE = rza \div p(z + b) - r = (rza - rp(z + b)) \div p(z + b)$.

Again, the Δs CLG, TSQ, being similar, it will be as $LG : CG :: SQ : QT = (ryza - rpy(z + b)) \div qza$; therefore $x = CT - QT = CT - (ryza - rpy(z + b)) \div qza$.

But, by sim. Δs $LE : EF :: BR : FE + ER$,
and $EF : EK :: ER - EF : AP$;
ther. com. $LE : EK :: BR \times (ER - EF) : AP \times (FE + ER)$;
but $LE : EK :: EC : ET$;
theref. $EC : ET :: BR \times (ER - EF) : AP \times (FE + ER)$;

hence, $ET = \frac{rc(b+z)}{a(b-z)}$ and $CT = r - \frac{rc(b+z)}{a(b-z)}$;

theref. $x = r - \frac{rc(b+z)}{a(b-z)} - \frac{ryza - rpy(b+z)}{qza}$.

Now $FH = (n \div m) \times EF$, $EH = (r \div m) \times EF$;

hence, $CH = r - (r \div m) \times EF = r(m-z) \div m$.

And by sim. Δs , $FH : HC :: SQ : QC$,

or, $(n \div m)z : r(m-z) \div m :: y : x$;

therefore $z = ymr \div (nx + yr)$. This value of z being substituted in the value of x found above,

$$\text{gives } x = r - \frac{rc}{a} \times \frac{bnx + byr + ymr}{bnx + byr - ymr} \\ - \frac{ryam + p(bnx + byr + ymr)}{qm}$$

the equation required.

And thus the answer is given by Messrs. Lowry, Swale, and Thornoby.

XII. QUESTION 80, answered by Mr. William Adamson, of Lower Belgrave Place, Pimlico, London, late Teacher of the Mathematics at Bridlington, in Yorkshire.

1. Having drawn a circle with the chord of 60° , make the arch AB (fig. 210, pl. 15,) equal to the time between the observations in degrees, and draw the diameter Gd, parallel to a line passing through A, B.

2. Make $de =$ the sun's declination; and with the cosine Cx, and one foot in the centre C, draw the semicircle xba.—From the points A and B towards the centre C, draw Aa, Bb to the inner circle, and through these two points draw a right line.

3. With the chord of the first co-altitude and one foot in a, draw an arch, and with the chord of the second co-altitude and one foot in b, cross that arch in D—draw DEF perpendicular to the diameter, and set the sine of the declination xe towards the centre C from G to H.

4. With the radius ED and one foot in E, draw an arch, and with radius EG and foot in H, cross that arch in p, and draw pP parallel to the diameter. Then CPM being drawn, will represent the meridian of the place, and CP the cosine of the latitude.

Scholium. When the observations are one before, and the other afternoon; the arch Am turned into time, will shew how long before noon the first observation was taken; and, in like manner the arch Bm will shew the time of the second observation afternoon; and consequently, when both observations are before noon, the point m will fall on the contrary side of B, and when they are taken afternoon, on the contrary side of A.

Note also, that when the sun's declination and latitude of the place are of contrary names, that is, one north and the other south, the points p and H must fall on the same side of the line DF, as in this

this example; but when the latitude and declination are of different names, they must be taken on contrary sides of DF.

The same by Mr. John Dawes, Surgeon.

Conceive a figure in which Z represents the zenith, P the pole, A and B the sun's places at the times of observation, Then in the isosceles triangle ABP there is given $AP = BP =$ the co-declination, and the angle $APB =$ the elapsed time; therefore the triangle ABP may be projected by art. 211, Crakelt's Trigonometry; and then AB becomes known. Then in the triangle AZB the three sides are given (AZ, BZ being $=$ the co-altitudes) and the triangle AZB may be projected without drawing ellipses by art. 209, *ibid.*

A true and ingenious solution was also given by Mr. Lowry.

XIII. QUESTION 81, answered by Mr. Gregory.

In fig. 211, pl. 15, let A and B represent the situations of the tacks, placed on a line making an angle (BAH) of 45° with the horizontal line AH, and let C be the place of the ring at rest. Then (supposing the string without weight) by the nature of forces, the angles ACD and BCF are equal, when the ring is at rest; whence this pleasing question may be solved by the following algebraical process. Put $AC = y$, $DC = x$, $DF = AH = BH = AB \times \text{nat. sine of } BAH = 2\sqrt{\frac{1}{2}} = b$, and $AC + CB = 5$ (per question) $= c$. Then by Trigonometry, as $y : 1 :: x : x \div y = \sin. DAC$; and, as $c - y : 1 :: b - x : (b - x) \div (c - y) = \sin. CBF$; but DAC and CBF are equal, because their complements are; therefore $x \div y = (b - x) \div (c - y)$; hence $cx = by$, and $y : x :: c : b$ the ratio of AC to CD: therefore the nat. sine of $CAD = CBF = b \div c$ and their cosine is $= \sqrt{1 - b^2 \div c^2}$. Again by Trigo-

Trigonometry, $\sin DAC : DC : \sin ACD : AD$
 $= (cx \div b) \times \sqrt{1-b^2 \div c^2} : \text{but } (c \div b) \times \sqrt{1-b^2 \div c^2}$
 $= 2.134374$, let this be denoted by d , and then
 $AD = dx$; also $\sin CBF : CB :: \sin BCF : BF =$
 $(b-x) \times (c \div b) \times \sqrt{1-b^2 \div c^2} = (b-x)d$. But it is
evident, that $BF - AD = BH = DF$; hence this
equation, $(b-x)d - dx = b$, that is, $db - 2dx = b$, from
which we readily get $x = (bd - b) \div 2d = .5637181$
 $= DC$, and $b - x = 3\sqrt{\frac{1}{2}} - .5637181 = 1.5576023$
 $= CF$. And by $\sin \Delta s$ we have $DC + CF :$
 $AC + CB :: DC : AC :: CF : CB$ —hence AC and
 CB are 1.327 and 3.673 respectively; thus the
position of the string is determined.

But this question may be answered without algebra, in the following manner. The two tacks A and B to which the string is fixed will be the foci of the ellipse ICL which will be described by the motion of the ring along the string when kept tight. The ring will rest in equilibrio when it is at the lowest point to which it can arrive; which will be at the point C of the ellipse, where the tangent GCF is parallel to the horizon. Let E be the centre of the ellipse, and let lines EF and ED , be drawn from it to meet the points F and D where, perpendiculars from the foci B and A , fall upon the tangent. Then by *Hutton's Conics*, pages 25, 26, 27. $ACD = BCF$, $ED = EF = EL$ the semitransverse; also ED parallel to BC ; and EF parallel to AC . Since $AGD = EAa = 45^\circ$, it is evident $EAD = 135^\circ$: hence, as $\frac{1}{2}(AC + CB) = EL = 2.5$, we have, $ED(2.5) : \sin. EAD (135^\circ) :: AE(1.5) : \sin. ADE = 25^\circ 6' \frac{1}{4}$, whence AED is found $= 19^\circ 53' \frac{1}{4} = ABC$ also, as is obvious. If Bb be parallel to GF , then $bBC = BCF$: hence $bBA (45^\circ) + ABC = 64^\circ 53' \frac{1}{4} = BCF = ACD = CAa$; therefore $EAa(45^\circ) + CAa = 109^\circ 53' \frac{1}{4} = BAC$. Then $180^\circ - (BAC + ABC) = 50^\circ 12' \frac{1}{2} = ACB$.

We have now given each angle of the triangle ACB , and the side $AB = 3$, to find the other two sides,

sides, which are $AC=1.328$ and $CB=3.672$: agreeing nearly with the algebraical result, and would have corresponded exactly, if the angle ADE, and those depending upon it, had been determined with sufficient accuracy.

Or, when ADE was determined, we might have proceeded in a different manner to arrive at the same conclusion. Thus, ADE, and EAD being known, AED is known also; theref. AD may be readily found, AGD being $=45^\circ$, $AD=DG$; hence GA may be determined. As the triangles GAC and GEF are similar, also GED and GBC, AC and BC may be easily discovered by the following proportions:

$$GE : EF :: GA : AC;$$

$$GE : ED :: GB : BC*.$$

We cannot close this solution without expressing our regret at the loss of the ingenious proposer of this question, Mr. BUCHANAN; who has *paid the debt of nature!* By his death the Sciences are deprived of a steady advocate in their cause, and the Editors of various periodical publications, of an able correspondent.

The same answered by Mr. J. H. Swale, Leeds,

Who, proceeding on the same principles, determines the point C in the following manner, produce BC and AD 'till they meet in Q; then by reason of the $=\angle$ s, $QC=AC$, and $\angle CAD=CQD$. Hence there is given $AB=3$, $BQ=5$, and $\angle BAQ=135^\circ$ to find $\angle BQA=25^\circ 6'$. Whence $\angle BCA=50^\circ 12'$; $CBA=19^\circ 54'$, and $BAC=109^\circ 54'$. Consequently $BC=3.67$, and $AC(CQ)=1.33$ feet. *Elegant Solutions to this Question were also received from Messrs. Buchanan, Evans, Lee, Lowry and Simpson.*

* Or, ADE having been determined as before $=HBA=25^\circ 6'$. Then, since $BH=3\sqrt{\frac{1}{2}}$, Ba is found $=2.346$. But by reason of the equal angles and parallels, $AC=aC$; therefore $2AC=AC+CB-Ba=5-2.346=2.654$, hence $AC=1.327$ and $BC=3.673$.

END.

XIV.

*XIV. QUESTION 82, answered by Mr. J. Lowry,
Officer of Excise, Birmingham.*

Conf. Let O (fig. 212, pl. 15.) be the centre of the circle of which BMC is a segment; take $AQ = 2AL - 2AO$, and thereon describe the semicircle QRA. At C erect the $\perp CP$, and let it meet the quadrant in P; in AD produced take $AZ = CP$, and draw ZR parallel to OAE meeting the semicircle QRA in R. Through R parallel to AC, draw RSM to meet the segment in M, and through M draw the required radius AML.

Demon. Let OC, OM be joined.

Then by *Conf.* $(2AL - 2AO - MN) \cdot MN = (AQ - MN) \cdot MN = QSA = RS^2 = CP^2 = AL^2 - AC^2$.

Therof. $AL^2 - 2AL \cdot MN + MN^2 = AC^2 - 2OA \cdot MN$,
or, $(AL - MN)^2 = AC^2 - 2OA \cdot MN$.

But by *Simp. Geo. II. 10* $(AL - ML)^2 = AC^2 - 2OA \cdot MN$,
that is, $AM^2 = AC^2 - 2OA \cdot MN$;

therefore, $(AL - MN)^2 = (AL - ML)^2$,
consequently $MN = ML$. *Q. E. D.*

Remark 1. If the points C and D coincide, $2AL - 2AO = MN$.

2. If C falls on the contrary side of D, the construction will be effected in nearly the same manner.

The same answered by Mr. Swale.

Analysis. Suppose the thing done, and O the centre of the circle of which the given segment is a part; in AO produced take $AT = AE$; join OM, OD, and demit upon BO, the $\perp MS$.

Then by *Emerson's Geo. II. 22*. $OM^2 = OA^2 + AM^2 + 2OAS$: but $OM = OB$ is given, and $AM = AL - ML = AL - MN = AL - AS$; hence, $OA^2 + AL^2 + AS^2 + 2OAS - 2LAS = OM^2$. Consequently $(2OT \propto AS) \cdot AS = OD^2 \propto OM^2$. Hence this construction. Take any line $TV = OT$, and divide it in F, so that the rectangle TFV may

$be = OD^2 \div OM^2$; then is $FV = AS$, and the point M is determined by drawing SM parallel to AD , as is evident from the analysis.

XV. QUESTION 83, answered by Mr. Lowry.

Let APB (fig. 213, pl. 15.) be the primitive circle. Draw the right circle AOB , and lay off BC, BD , equal the given sides respectively; about the point C as a pole, with distances CB, CD , describe the lesser circles BR, DP , the latter intersecting the primitive in P . Through P draw the great circle PR to touch the lesser circle BR in R , and PCR will be the triangle to be projected.

For, the angle at R is a right one, $CP = CD$, and $CR = CB$, and therefore $PR = DB$.

Mr. *Robertson*, at page 122 of his *Treatise on Mathematical Instruments*, has given a very ingenious and general construction of this problem to the following effect:

On the right line ed (fig. 214, pl. 15.) touching the primitive circle at any point b , take $db, be =$ the tangents of the given sides; make the $\angle dba =$ the given one; and take $ba = be$. From d and O , with distances da, Oe , describe arches crossing in x ; draw $dDOF, xAOE$, and from d and x , with distances db and eb , describe arches crossing in B . Through the points $A, B; D, B$ describe great circles, then is ADB the triangle required.

Since $Ox = Oe$, therefore $Bx (= be)$ is a tangent to the circle ABE at B , also $Bd (= bd)$ is tangent to the circle at B ; therefore BD and AB are equal the given sides. Moreover the triangles Bdx, dba having the three sides equal, have also $\angle dBx = dba$; therefore $\angle DBA = dba =$ the given angle.

This question was also answered by Mr. Swale.

XVI. QUESTION 84, answered by Mr. Harris.

Case 1. When the sum of the greater side and the adjacent segment of the base is given.

From

From any point B (fig. 191, pl. 13.) in the indefinite right line AD, draw BE making the $\angle CBE = \text{comp. of } \frac{1}{2} \text{ the given diff. of the } \angle\text{s at the base.}$ Take $BE = \text{the given bisecting line, and } BA = \text{the given sum.}$ Join AE, and make the $\angle OEA = OAE$, and draw EO intersecting AD in O; then make the $\angle BEC = BEO$, and draw EC meeting AD in C; so shall OEC be the Δ required.

For, by *Conf.* the angle $OEB = BEC$.
And (Prop. II. p. 284) $\angle EBC = \text{comp. } \frac{1}{2}(\angle B - \angle A)$, also by *Conf.* $\angle OEA = \angle A$; theref. $AO = OE$; and theref. $OB + OE = OB + AO = AB$, the given sum.

Case 2. Draw the line BE, &c. as in the 1st case, and take $BD = \text{the given difference.}$ Join ED, and make the $\angle DEO = ODE$, and draw EO meeting AD in O; make the $\angle BEC = BEO$, and draw EC, so shall OEC be the Δ required.

Elegant constructions were also received from Messrs. Lee, Lowry and Swale.

XVII. QUESTION 85, answered by Mr. Swale.

Conf. Draw the great circles AB, BQ (fig. 73, pl. 4.) to contain the given vertical angle, and take BP, BQ each $= \text{half the given sum of the sides.}$ Draw the perpendicular arches QD, PD, to intersect in D, and draw DC to make the $\angle DCQ = \text{the complement of half the given diff. of the } \angle\text{s at the base.}$ Make $AP = CQ$, and through the points A and C describe a great circle, and ABC will be the Δ required, as is evident from Prop. XIV. (art. XI. Rep.) of *Lowry's Spherical Lucubrations*, and its corollaries.

Mr. John Lowry, after giving the solution exactly in the same way, solves the problem as follows, by constructing the supplemental Δ , in which there is given the base, the diff. of the $\angle\text{s at the base, and the difference of the sides.}$

In fig. 73, pl. 4, let AC be taken $= \text{the given base, and the arches CD, AD be drawn to make the}$

\angle s ACD, CAD each $= \frac{1}{2}$ the supplement of the given sum of the \angle s at the base. Construct the Δ s DCQ, ADP, right angled at Q and P; and having CQ, AP, each $= \frac{1}{2}$ the given diff. of the sides; continue AP, CP 'till they meet in B, and ACB will be a triangle supplemental to the required one. For, Prop. XIV. art. XI. Cor. 1, \angle BAD + BCD = 180° or, \angle BCA + BAC + 2ACD = 180° ; but the given sum of the \angle s = $180^\circ - 2ACD$; therefore the \angle BCA + BAC = the given sum. Moreover, it is evident that PB is = BQ; therefore AB - BC = 2QC = 2AP = the given diff.

XVIII. QUESTION, 86, answered by Mr. Sanderson, London.

If the Δ ACB, fig. 215, pl. 15, be circumscribed by the circle ACBE, and CE be the given line bisecting the given angle ACB; and from the point E, the line ED be drawn at right angles to AC: then

$$CD = \frac{CA + CB}{2}, \text{ and } AD = \frac{AC - CB}{2}; \text{ but CE,}$$

and the angle ACE, are given, therefore CD is

$$\text{given, and } \overline{CD + AD}^3 (CA^3) + \overline{CD - AD}^3 (CB^3)$$

$$= c^3, \text{ the given cube per question. Or } 2CD^3 + 6CD \times AD^2 = c^3; \text{ whence, } 2CD : c :: c^2 : CD^2 + 3AD^2 :: 2CD \times c : c^2; \text{ it remains to find AD.}$$

Make CF = c the side of the given cube; and CG a mean proportional between 2CD and c; CI a third proportional to CG and CF: then $CI^2 = CD^2 + 3AD^2$, and $CI^2 - CD^2 = 3AD^2$: or

$$\frac{CI + CD}{3} \times \overline{CI - CD} (ID) = AD^2, \text{ whence half the}$$

difference of the sides (AD) is a mean proportional

$$\text{between } \frac{CI + CD}{3} \text{ and ID.}$$

Observation. The next question may be constructed in the same manner, as it is reducible to the sum, and the sum of the cubes as above.

The

The same answered by Mr. Swale.

Analysis. Fig. 175, pl. 12. Suppose the problem solved, and ACB the required Δ , circumscribed by the circle ACBE, the diameter of which bisects the base AB. Demit upon AC the \perp EP, and join EA, EB.

Since CE and the $\angle ECP$ are given, CP will be given; hence $AC + CB = 2CP$ will be given.

Again, $AC^3 + CB^3 = (AC^2 + CB^2 - ACB) \times (AC + CB)$.

But $AC^2 + CB^2 = (AC + CB)^2 - 2ACB = 4CP^2 - 2ACB$; therefore, $AC^3 + CB^3 = (4CP^2 - 2ACB) \times 2PC = M^3$, or, $4CP^2 - 2ACB = (M \div 2PC) \times M^2 = Q^2$, taking $Q^2 : M^2 :: M : 2PC$. Therefore $2ACB = 4PC^2 - Q^2$; hence, the problem is reduced to this, viz. To divide the right line AB into two parts AC, CB, so that the rectangle ACB may be $=$ to a given magnitude; which is a well-known problem.

This question was also answered by Messrs. Lowry and Peletarius.

XIX. QUESTION 87, answered by Mr. Lowry.

Conf. Let ABCD (fig. 216, pl. 15,) be the given square, AC its diagonal, which bisect in O, and demit OQ \perp to AB; produce AB till AR $:: M^2 :: M : OQ$ (M being $=$ to the side of the given cube.) To CB produced apply AL $=$ AR. On the diagonal take OD; so, that $6OD^2$ may be $= BL^2$, and D will be the point required.

Demon. Demit the perpendiculars DE, DF, DG, and DH.

By *Conf.* $6OD^2 = BL^2 = AL^2 - AB^2$:

but $AL^2 = LR^2 = M^3 \div OQ$;

therefore $6OD^2 = M^3 \div OQ - AB^2$:

but AB is equal to 2OQ,

therefore $6OD^2 + 4OQ^2 = M^3 \div OQ$,

or, $4OQ^3 + 6OD^2 \cdot OQ = M^3$.

E 3,

But

But O is evidently the centre, and OQ the radius of a circle inscribed in the given square, therefore by Prop. XXIII. of Dr. Stewart's *General Theorems* (which will be demonstrated in its proper place).

$DE^3 + DF^3 + DG^3 + DH^3 = 4OQ^3 + 6OD^2 \cdot OQ$,
therefore $DE^3 + DF^3 + DG^3 + DH^3 = M^3 =$ the given cube.

The *locus* of the point D is evidently a circle, whose centre is O and radius OD; consequently there will be four points in the diagonal of the given square which will satisfy the conditions of the problem.

The sum of the cubes will evidently be a maximum when D coincides with C, and a minimum when it coincides with O.

The same answered by Mr. Swale.

Analysis. Let ABCD be the given square, D the point required, and DE, DF, DG, DH, the perpendiculars.

Then $DF = BE = DG$, and $DH = CE = DE$;
theref. $DF^3 + DE^3 + DH^3 + DG^3 = 2DF^3 + 2DE^3$
 $= 2BE^3 + 2CE^3 = M^3 =$ a given cube.

But it appears, as in the last question, that $(BC^2 - 3BEC) \times 2BC = 2BE^3 + 2CE^3 = M^3$;

theref. $BC^2 - 3BEC = (M \div 2BC) \times M^2 = Q^2$,

taking $Q^2 : M^2 :: M : 2BC$;

hence $3BEC = BC^2 - Q^2$;

therefore BC and the rectangle BEC are given to find BE, EC; the method of doing which is well known.

XX. Or PRIZE QUESTION 88, answered by the proposer, Mr. William Wallace.

Let DC, DE, CE (fig. 217, pl. 15,) be three tangents to a parabola, a circle described about the triangle DCE shall pass through F, the focus of the parabola.

Let

Let V be the vertex, and let FL be drawn perpendicular to LM the directrix; let FG be drawn perpendicular to DC any of the tangents, meeting the directrix in M , join VG .

From the nature of the parabola FG is equal to GM , and FV is equal to VL , therefore GV is parallel to ML , hence GV is a tangent to the curve at the vertex, and in like manner it will appear that if FH , FK be drawn perpendicular to the tangents DE , EC , the lines joining the vertex and the points H , K , also touch the curve at the vertex; hence a tangent at the vertex passes through the points G , H , K . Join FD , FC , FE . Because FGC , FKC are right angles, the points F , G , C , K are in the circumference of a circle, hence the angle FCK is equal to FGK , that is, FCE is equal to FGH : but FGH is equal to FDH or FDE , for, the angles FHD , FGD being right angles, the points F , D , G , H are in the circumference of a circle, therefore FCE is equal to FDE ; hence the points F , D , C , E are in the circumference of a circle, or if a circle be described about the triangle DCE , it will pass through F , the focus of the parabola, as was to be demonstrated.

Cor. 1. If CD , CE touch the curve at A and B , and if FA , FB be joined, the triangles AFC , CFB are similar to the triangle DFL .

For if AM be joined; from the nature of the curve AM is parallel to FV , and the angle AFG is equal to AMG , or to GFV , now AGF , GVF are right angles, therefore the triangles AGF , GVF are similar, and the angle FAG is equal to FGV , or FGH , that is, because the points F , D , G , H are in a circle to FDG ; now the angle FCG is equal to FEH , therefore the triangles FAC , FDE are similar. In the same way it may be shewn that the triangle FCB is similar to FDE .

Cor. 2. Let AC , BC , be two straight lines given by position touching a given parabola, if any other tangent be drawn meeting them in D and E , and if FD , FE be joined, the triangle FDE is given in species. For it is similar to each of the given triangles AFC , CFB .

Cor. 3. If AC , BC are two tangents to a parabola, and if a circle be described through the focus to touch BC , one of them at C , it shall pass through A , the point where the other tangent meets the curve.

For it has been shewn that the angle FAC is equal to FCB .

Cor. 4. If AC , BC , DE are tangents to a parabola at the points A , B , N ; these lines are similarly divided at their intersections and the points of contact, that is, $AD:DC = DN:NE = CE:EB$.

For by *Cor. 1*, the triangles AFC , CFB , DFE are similar, and if FN be joined, it will appear in like manner that the triangles AFD , DFN , CFE are similar, and that NFE , EFB , DFC are similar

similar, therefore $AD : DF = DN : NF$
 and $DF : DC = NF : NE$,
 hence $AD : DC = DN : NE$;
 and in the same way it appears that $DN : NE = CE : EB$.

The proposition contained in this last corollary appears to have been first observed by Dr. Halley, in his Translation of the *Sectionis Rationis* of Apollonius, and it may be found in most books on the conic sections.

From the preceding proposition and its corollaries may be derived easy solutions to certain problems respecting the description of parabolas that may pass through given points and touch straight lines given by position.

The same answered by Mr. Lowry.

Let AC, BC, DE, be three tangents to the parabola AVB, intersecting each other in C, D, and E, and let F be the focus, V the vertex, and GVK a tangent to the parabola at V. Draw FG, FH, FK, which, by prop. 12, book III, Emerson's Conics, will be perpendicular to AC, DE, and BE respectively;

therefore the points F, H, E, K, are in a circle,

and so are also the points F, G, C, K;

therefore the $\angle FCG$ or $\angle FCD = \angle FKG = \angle FEH$ or $\angle SEF$;

but the $\angle DSC = \angle FSE$; therefore the \triangle s DSC, FSE, are equiangular; therefore, the rect. DSE = rect. CSF; wherefore the points C, D, E, F, are in a circle.

Mr. Robert Wallace, Teacher of the Mathematics, at Newcastle-upon-Tyne, after giving a solution exactly the same as Mr. Lowry, above, adds the following Corollaries:

1. $GY : YF :: CY : to YK$, Y being the intersection of GK, CF.

2. $GK \cdot CF = GC \cdot FK = GF \cdot CK$.

3. $\angle KCW = \angle KFG$; and $\angle FKC + \angle FGC = \angle GFK + \angle GCK$.

4. If there be given four tangents to a parabola, viz. DE, AC, BC, GK; to find the focus and describe the parabola.

It is evident, from the proposition, that the circles which circumscribe the \triangle s DCE, GCK, will intersect each other in the focus F; which being found, the parabola may be described by the known methods.

5. If it be required to draw a tangent from a point A in a parabola; or from a point X without it.

Join FA, or FX, and draw GVK a tangent at the vertex. Upon FA, or FX, describe a semicircle intersecting GK in G, through which draw AGX, the tangent required.

6. If there be given three tangents AC, BC, GK, and the ratio of GF : FK; to find the focus and describe the parabola.

Because the points G, F, K, C, are in a circle, the $\angle GFK$ is given,

given, and GK is given in magnitude, therefore the $\triangle GFK$ is given in magnitude and species; hence, the point F, which is the focus, becomes known, and the parabola may be described.

Ingenious solutions were received from Messrs. Lee, Sanderson, and Swale.

The Medal for solving the Mathematical Prize Question is decided in favour of Mr. William Wallace, who will please to send for it to Mr. Glendinning's, by whom it will be delivered, free from expence, to any part in London.

ARTICLE X.

MATHEMATICAL QUESTIONS,

(To be answered in Number VIII.)

I. QUESTION 110, by Mr. Newton Bosworth, Teacher of the Mathematics at Peterborough.

IF a cubic piece of fir sink 6 inches in water, how far will it be immersed in oil of turpentine, and what are its dimensions, allowing the specific gravities of water, oil of turpentine, and fir to be 1000, 800, and 550, respectively?

II. QUESTION 111, by Mr. O. G. Gregory.

A neighbour of mine has a cask in form of a conic frustum, the altitude of which is 28 inches, and the diameter of the bases 18 and 14 inches respectively; in each end is a cork hole one inch in diameter. As this cask is filled with excellent Ootober, and my neighbour is fearful of losing it by one of the corks flying out, he wishes to be informed which end of the vessel should be set downwards, that the liquor may run out most slowly if an accident should happen; and in how much longer time the cask would be emptied with one end at the bottom than with the other?

III. QUESTION 112, by Mr. Newton Bosworth.

There is a cylindric vessel, kept constantly full of water, which stands upon a wall 9 feet high; and at two feet from the bottom of the vessel there is a small orifice through which the water spouts and falls into another vessel on the level ground, at the distance of 14 feet from the wall. It is requested that some of the ingenious Correspondents to the Mathematical Repository will, from these data, determine the altitude of the vessel?

IV.

IV. QUESTION 113, by Mr. Harris, Caermarthen.

A round piece of timber, whose girths at the ends are 9 and 2 feet, was measured by the common method. Now if it be cut into two pieces at the distance of 12 feet from the lesser end, and both the pieces measured according to the same method, they will amount to three solid feet more than the whole piece; required the length of the said piece of timber?

V. QUESTION 114, by the Rev. Mr. L. Evans.

Bought four gallons of brandy of A, at 18 shillings per gallon, which by *Quin's* hydrometer was found to contain 10 gallons of water in 100 of proof spirits. Bought also the same quantity of B, at 10 shillings and sixpence per gallon, which contained 18 in 100. I demand how much per gallon was one cheaper than the other?

VI. QUESTION 115, by Mr. Sanderfon, London.

D, E, and F, are three points in the same right line; E is distant from D 300 yards; F is distant from D 400, and from E 100 yards. Three men, A, B, and C, start at the same time from D, E, and F, and meet in a fourth point G. Now the distance run by A (DG) is to the distance run by B (EG) as the distance run by C (FG) is to the distance DF; and the $\angle DGE = 2EGF$, required the distances run by A, B, and C, without algebra?

VII. QUESTION 116, by Mr. Robert Wallace.

Let CA, CB, be the asymptotes of a given hyperbola; it is required to draw the straight line AB touching the hyperbola, so that the part of it AB, intercepted by the asymptotes, may be of a given length?

VIII. QUESTION 117, by Mr. J. H. Swale, Leeds.

Given the line drawn from the vertex and bisecting the base, and either of the \angle s at the base, to construct the Δ when the area is a maximum.

IX. QUESTION 118, by Mr. Swale.

A ball of copper, 12 inches in diameter, after rolling down a smooth plane (whose length was 60 feet, and inclination to the horizon 40°) fell into a reservoir full of rain water: now suppose the fluid to have destroyed all the globe's velocity precisely by the time it arrived at the bottom: it is required to determine the depth of the reservoir; the distance of the lowest point of the plane and the surface of the water being 45 feet, and supposing the globe to descend perpendicularly in the fluid?

X.

X. QUESTION 119, *by Mr. Swale.*

Given the longest side, and the difference of the segments of the base made by the \perp to construct the \triangle , when the rectangle under the given side and the less segment of the base, has to the rectangle under the \perp and the other side a given ratio.

XI. QUESTION 120, *by Mr. Ralph Simpson.*

A glass in form of a paraboloid 4 inches deep being filled with wine, and its axis inclined so as to make an angle of 36° with the plane of the horizon; it was observed that half the wine just run out. How much did the glass hold when full?

XII. QUESTION 121, *by Mr. Thomas Bulmer.*

A weight of 4lb. being dropped from a certain height, and striking against a spring, was found to bend it three inches, the elastic force of the spring so bent being $= 1000$ lb. From what height did the weight fall, and how long was it in descending?

XIII. QUESTION 122, *by Mr. William Peacock.*

To what height must Telegraphs be erected at Berwick-upon-Tweed, Dover, in Kent, and Penzance, in Cornwall, so as to be able to communicate with another Telegraph 300 feet high, erected at Birmingham, in lat. $52^\circ 30'$ N. and long. $1^\circ 50'$ W.

Note. The lat. of Berwick is $55^\circ 48'$ N. and long. $1^\circ 45'$ W.; lat. of Dover $51^\circ 7'$ N. and long. $1^\circ 13'$ E.; lat. of Penzance $50^\circ 8'$ N. and long. 6° W.; and the diameter of the earth 7960 miles.

XIV. QUESTION 123, *by Mr. John Johnston.*

On an horizontal plane are placed two globes whose diameters are 12 and 8 inches, and the distance of their points of contact with the plane 10 feet. Whereabouts in the line joining their points of contact must a lighted candle 13 inches high be placed, so that the spaces made by the shadows of the two globes may be equal?

XV. QUESTION 124, *by Mr. Hewitt, Teacher of the Mathematics, Bunhill-Row, London.*

What must be the weight of an equilateral leaden cone, so that, being immersed with its base downwards in a glass goblet, filled with water, in the form of a paraboloid whose altitude is 4, and diameter at top 3 inches, the greatest quantity of water may overflow?

XVI. QUESTION 125, *by Mr. Francis, Jun. of Hungerford.*

A globe 15 inches diameter being placed in water sunk so deep as to conceal $\frac{2}{3}$ of its surface; what was its weight? and what was the diameter of a globe of copper, which being connected to it by a string of no weight, sunk it just beneath the surface of the water?

XVII. QUESTION 126, *by Mr. Lowry.*

Given the vertical angle, the radius of the inscribed circle, and the sum of the distances from the centre of that circle to the angular points of the triangle to construct it.

XVIII. QUESTION 127, *by Mydorgius.*

If two polygons $a, b, c, d, e, \&c.$ and $p, q, r, s, t, \&c.$ be inscribed in an ellipsis, and through the respective points $a, b, c, d, e, \&c.$ and $p, q, r, s, t, \&c.$ the tangents $AB, BC, CD, DE, \&c.$ and $PQ, QR, RS, ST, \&c.$ be drawn, and the $\angle abB = \angle cbC$; $\angle bcC = \angle dcC$; $\angle cdD = \angle edE$; $\&c.$ and $\angle pQ = \angle rQ$; $\angle qrR = \angle srS$; $\angle rsS = \angle tsT, \&c.$ Then $ab + bc + cd + de + \&c.$ is $= pq + qr + rs + st + \&c.$ Required the demonstration?

XIX. QUESTION 128, *by Mr. Wm. Davis, London.*

It is required to find to what height a building may be raised at any given latitude, before the stones are carried off its top, by reason of the centrifugal force overcoming the centripetal?

XX. QUESTION 129, *by Mr. Lowry.*

In a given circle to inscribe a triangle, so that its sides may pass through three given points, not in the same straight line.

XXI. PRIZE QUESTION 130, *by Theodosius.*

If the ends of a thread of a given length be fastened to two given points on the surface of a given sphere; it is required to determine the nature of the curve made on its surface by moving a pencil round with the thread, keeping it always stretched close to the surface of the sphere?

ARTICLE XI.

*Reply to Mr. Lowry's Animadversions on Mr.
Howard's Spherical Geometry.*

To the EDITOR of the REPOSITORY.

SIR,

RELYING on that impartiality with which the Mathematical Repository has hitherto been conducted, I request permission to reply to Art. LX. Vol. I. of your ingenious work, entitled "*Animadversions on some remarks in the preface to my Book. By Mr. John Lowry.*"

To the first charge I had, previous to this attack, pleaded guilty, by publicly acknowledging the error, in an Appendix to my Book, sent to Mr. *Longman*, the beginning of June last; and which also, I presume, may be found in most of the latter issued copies.

Mr. *Lowry* next asserts that "the reasoning in several other of my theorems will, on examination, be found equally fallacious and absurd, and by dividing them into their particular cases, numerous errors will be discovered, which are at present concealed in each common mass."

To the former part of this remark, I can only reply, that assertions are not proofs, and, until Mr. *L.* brings forward something more, that neither the world nor I can be justified in assenting to his mere *Ipse dixit*. And with regard to the latter part, I did not know, until now, that the failure of a general theorem in a particular instance, came under the denomination of an error. Mr. *L.* here passes an eulogy upon his own conduct, which seems not easily reconcileable either with what precedes or

immediately follows it. This behaviour appears some what like that of a guest, at an ordinary, who assures the company that many of the dishes on the table are unwholesome, and yet declines to point them out. I need not say what the consequence would be to the host.

For the next charge, it will be sufficient to compare the paragraph there hinted at by Mr. L. with his paraphrase upon it, in order to shew his correctness and candour in quotation, his clearness in apprehension, and elegance of composition. The expression in particular "to whose confidence he communicated his secrets" has a grace beyond the reach of art, not easily, I think to be comprehended; which might for ever have remained in dark obscurity, but for the sublime conception of Mr. Lowry.

To explain and confute misrepresentation and absurdity, when thus jumbled together, were no easy task. Leaving Mr. L. therefore, to correct his own blunders, I shall explain, by a simple statement of facts, the purport of my note; and then, perhaps, he will find an answer to his quære respecting what discoveries I lay claim to, and who it is that I blame for having betrayed my confidence.

In a Letter from Mr. L. dated *Jan. 22, 1794*, after largely acknowledging favours received, and that the knowledge of Mathematics which he possessed he owed almost entirely to me, he adds, "I am in great hopes of seeing the publication of your Spherics and other new matters. If a sufficient number of subscribers cannot be procured, I would recommend publishing them in Numbers, with Essays, Quæries, &c. I think there would be little doubt of their success that away: ———'s publications (works infinitely inferior to your's) are a proof that works of this kind meet with success."

Who

Who that reads this would believe that the writer of it had already done, in his own name, what he had here so strenuously advised me to do? And yet (if there is faith to be put in dates) this was really the case. His packet containing three Spherical Problems, inserted in No. 7, Scientific Receptacle, founded upon the principles contained in my Treatise *, though published the April following, must have been sent to Mr. *Whiting*, at least, in the December antecedent!

That complaint in such a case, was justifiable, the world must allow; and that a confession of error, and a discontinuation of the offence, must have been the result of such complaint (by an honest man, at least) must also be allowed. I received no such satisfaction; 'twas in vain I remonstrated against a behaviour so unprincipled, urged confidence betrayed, or pleaded the cause of injured fame. I did not even receive an answer to my Letter. A partial acknowledgement of hints communicated, in page 326, No. 9, of the Scientific Receptacle, was all the reparation made; though the offence has been uniformly repeated till all the store became exhausted.

The evil has, however, proved much less than might have been expected. From a barrenness of invention, he has (like a hound at fault) been continually retracing the same path; whilst the clumsy and ungeometrical mode of his demonstrations (evidently borrowed from Emerson's Trig. latter Eds.) with the injudicious order in which he has arranged his propositions, so far from inciting emulation, have hitherto even prevented competition.

Mr. *L's* invidious misinterpretation of the meaning of my note, which has given him so much

F 2

scope

* See, in particular, Quest. 153, which is C. 4, T. 7. B. 4. of my Treatise.

scope for animadversion, has also given me much additional trouble in confuting him. Obligated thereby to follow this *PROTEUS* through all his doublings and windings, though to as little purpose as a man would sift a bushel of chaff for a single grain of corn.

I now return back to where he says, "had I pointed out what discoveries I laid claim to, he should have been better able to give me an answer." It would hence appear that Mr. *L.* has not read my preface (much less the book), or he must have found that, independant of having digested a book of Elements, there is not a section in the whole where I do not, and I may add where I cannot, claim the merit of originality both in matter and manner.

"No," says Mr. *L.* "there are similar propositions in Euclid's Elements, Emerson's Geometry, &c. and the application of similar principles and Spherics was evident, natural, and easy."

The anecdote of the Egg, related by Columbus, was never more applicable than here. If the transition was so evident, &c. why were not these props. discovered before, and why did not Mr. *L.* who certainly has taken much trouble, bring them forth to light ere this time? Why did he solve P. 18, Art. Sph. in the case of maximum only, when it might have been done generally and geometrically? See P. 20, B. 1, of Con. Why did he propose Quest. 190, Scientific Receptacle, only partially, when Mr. *Armstrong* proved it was true in general*? Why too did he solve P. 21, Art. Sph. (and that by the ungeometrical mode of lines and tangents) only partially, when it admitted of a general geometrical solution? See P. 21, B. 1, of Con.

* Mr. *L.* has with his usual facility at manufacturing the hints of others, wrought the substance of that Demon. into a prop. (22) of his *Lucubrations*, but without being able to discern its important uses.

Con. And why did he solve P. 23, Ibid. partially, when he (now) knows that it also admits of a general solution? See P. 19, B. 1, of Con. Lastly, why did he not attempt P. 22, B. 1, of Con. at all? The truth is, he could find no hint in Simpson's Geometry, (where the same props. in plano are to be found) to lead him to their solutions, and his new method of solving Geometrical Problems by Trigonometry, could not apply here.

In answer to Mr. L's wish for a comparison between some Theorems in No. 3. Repository, and B. 4, of my Treatise, I refer him to the fable of the Frog and the Ox, informing him, at the same time, that there are propositions, as well as modes of demonstration, in that book not to be found in Simpson's Geometry, nor in any other work that I know of.

I am, Sir, With much respect,

Your obliged humble Servant,

Oct. 18, 1798.

JOHN HOWARD.

ARTICLE XII.

To the Editor of the Mathematical Repository.

SIR,

IN page 400, Vol. I. of your valuable work, there is a personal allusion to myself, on which I beg leave to trouble you with a few observations. The paragraph runs thus, " If the equation in the 20th question, No. 2, had been $\frac{bx - ab}{c} = y$; the

locus would have been the right line GH (fig. 237, pl. 16.) cutting the right line AE in the given angle ABH, where $AB = a$, $AD = x$, $FD = y$; and BD to DF in the given ratio of c to b ; or when bx is less than ab ; Bd to df in the given ratio of c to b .

But the author of a book lately published, has pronounced DF (y) impossible when bx is less than ab : his reason for this it is not difficult to perceive, as he has in several parts of the work, given abundant reason for presuming that he knows of no other use of the negative sign than that of subtracting a less quantity from a greater."

To the charge of knowing "no other use of the negative sign than that of subtracting a less quantity from a greater," I answer. *Habes confitentem reum.* I know no other use of it, but I know an abuse of it, namely, the applying of it so, that a term shall be understood to be less than nothing, which is nonsense. Indeed the writer of the paragraph, which I have quoted, is not accurate in his presumption, since for several years I abused the sign in the same way that he and many others do, talking consequently of algebraical terms, by rote, like a parrot; and being incapable through the infatuation, in which I had been educated at the University of Cambridge, of understanding, for a considerable time, the beautiful simplicity pervading the construction and theory of equations. It is difficult, I know, for persons, accustomed to the present unscientific mode of reasoning in Algebra, to come back to the pure principles from which they have deviated; but having had considerable experience in the modern, and not a little already in my own system, I have not the least doubt, that in the next generation impossible and negative roots will share the same fate with the vortices of Descartes, and the ultimately equals of Sir Isaac Newton.

I pronounce DF to be impossible, when bx is less than ab , for a very plain reason: namely, when bx is less than ab , the term $bx - ab$ gives no number at all. Consequently DF cannot exist in space, for it is impossible to assign any part or parts of a foot,
mile,

mile, or any other measure, for its length. This is plain to any one. $\frac{bx-ab}{c}=y$, put into words, is to be thus read, y is equal to a number of feet denoted by $\frac{bx-ab}{c}$. To find this number, take away ab from bx , and divide the remainder by c ; but, this is impossible, for I cannot take away ab from bx , when ab is greater than bx . The locus of the equation $\frac{bx-ab}{c}=y$, when $AD=x$, and $AB=a$, I pronounce to be BG, and the locus of the equation $\frac{ab-bx}{c}=y$, may be BH. Algebra cannot determine the position of the line BH, whether it should be above or below the line AB, and if the locus of this equation $\frac{bx \propto ab}{c}=y$ were required, the true answer is, that it will be either HG or IK, or IBG or HBK, according as the first ordinate is taken above or below the line; as you permit the ordinate to be taken on either side, or require that it should be always on the same side of the line AE.

I might excuse myself from saying any thing on the expression in Simpson's Fluxions

$$x - \frac{x + n^2 a^2 x^2 n^{-1}}{n-1} = y, \text{ given in page 399, by}$$

the same correspondent, because the remarks on it are intended for two anonymous writers in a magazine, and I must expose myself to the epithet of a dabbler in the science I pretend to teach. I am content, however, to be called a dabbler, and leave the

the " real Mathematicians" to overthrow, if they can, my examination of the proof given, that, when $x=0$, and $n=\frac{1}{2}$, y will be equal to $\frac{a^2}{2}$. The proof is thus given in page 399. " Multiply the equation by x , and we have $x^2 - \frac{x^2 + n^2 a^2 \times x^{2n}}{n-1} = xy$; make $n=\frac{1}{2}$, and it will be $x^2 - \frac{x^2 + \frac{1}{4}a^2 \times x}{\frac{1}{2}-1} = xy$, or $y = x - \frac{x + \frac{1}{4}a^2 \times 1}{-\frac{1}{2}}$, equal when $x=0$, to $-\frac{\frac{1}{4}a^2 \times 1}{-\frac{1}{2}} = \frac{a^2}{2}$ as before." When n is made equal to $\frac{1}{2}$, the equation becomes $x^2 - \frac{x^2 + \frac{1}{4}a^2 \times x}{\frac{1}{2}-1} = xy$. Now $\frac{1}{2}$ is less than one; therefore I cannot take one from $\frac{1}{2}$, consequently the next equation $x^2 - \frac{x^2 + \frac{1}{4}a^2 \times x}{-\frac{1}{2}} = xy$ is nonsense. Let us, however, pursue the reasoning, and to do it more easily, we will call the term $\frac{x^2 + \frac{1}{4}a^2 \times x}{-\frac{1}{2}}$ *hocus pocus*, and then the equation becomes $x^2 - \text{hocus pocus} = xy$. The next step is to divide by x , and consequently $x - \frac{\text{hocus pocus}}{x} = y$, which, when $x=0$, becomes nothing $-\frac{\text{hocus pocus}}{\text{nothing}}$ is equal to y ; but how nothing $-\frac{\text{hocus pocus}}{\text{nothing}}$ can make $\frac{a^2}{2}$ the " real Mathematicians" must determine. The real fact is, and a very obvious one it is, that, when $x=0$, the expression

pression $x - \frac{x + n^2 a^2 x^{2n-1}}{n-1}$ becomes nothing, take

away nothing, add nothing, that is nothing at all; and if a radius of curvature is equal to such an expression in such a state of x , that radius will also be nothing at all. It may be asked, then, whether I disallow of these expressions found, as it is termed, by fluxions. By no means. I am not so zealous an admirer of the ancients, as to insist severely on that rigid mode of demonstration, for which so many of the moderns seem to have lost the taste.

Fingere cinctutis non exaudita Cethegis

Continget dabiturque licentia sumpta pudenter.

Let *licentia* be taken *pudenter*, but science must not be degraded by *hocus pocus*.

The equation, then, $x - \frac{x + n^2 a^2 x^{2n-1}}{n-1} = y$, I

allow to be true, for what is called the radius of curvature, though I do not admit any thing to be true about Curves, which is not consistent with the sixteenth proposition of the third book of Euclid. But this expression is not true in all possible values of n , for the least acquaintance with the principles of Algebra, will enable any one to see that it is not true unless n is greater than one, and that when n is

less than one the equation is $\frac{x + n^2 a^2 x^{2n-1}}{1-n} - x = y$,

consequently when $n = \frac{1}{2}$, the equation is $\frac{x + \frac{1}{4} a^2}{\frac{1}{2}} - x = y$, or $2x + \frac{1}{2} a^2 - x = y$, or $x + \frac{1}{2} a^2 = y$.

Now if I could permit myself to use the language of the "real mathematicians" and say, let $x = 0$, and $y = \frac{1}{2} a^2$, then the radius at the supposed point would

would be one-half of a^2 , but Algebra allows not such an abuse of language. I cannot say that x is equal to nothing at the conclusion, any more than at the beginning, of a question, and if x could be called nothing at any period, then the radius of curvature of all curves, when x is equal to nothing, would also be nothing. Thus, substitute nothing

for x in the equations $x - \frac{x + n^2 a^2 x^{n-1}}{n-1} = y$ or

$\frac{x + n^2 a^2 x^{n-1}}{1-n} - x = y$, and whether n is equal to,

greater, or less than one, y must be nothing.

From the above your Correspondent will see that I am not to be frightened by his distinctions between dabblers and real mathematicians, and as the first part* of my Principles of Algebra does not please him I do not expect his approbation of the second part now in the press. For his beloved negative sign is throughout used only in one sense, that of taking away; the theory of equations is laid down in a direct manner; and the position, that an equation has as many roots as it has dimensions, is rejected, because it is built upon error. A small specimen of my plan I gave a short time ago, in a letter† to the Vice-Chancellor of Cambridge, when I was candidate for the Lucasian Professorship of Mathematics, and from that specimen some of your readers may perhaps be induced to labour with

* Principles of Algebra, by William Frend. Sold by Robinson, Paternoster-row, and White, Fleet-street.

† A Letter to the Vice-Chancellor of the University of Cambridge, by William Frend, price 6d. White, Fleet-street.

me in the same career, and, instead of troubling themselves with negative and impossible roots, assist in restoring Algebra to its true principles.

I remain, Sir,

Your's, &c.

WILLIAM FREND.

No. 4, Hare-court, Temple,
4th Dec. 1798.

ARTICLE XIII.

*Demonstrations to Lawſon's Propositions proposed in
ARTICLES XL. and LIII. Vol. I. PROP.
XXII. Fig. 218, Pl. 15, and Fig. 223,
Pl. 16.*

Demonſtrated by Mr. Colin Campbell, Kendal.

Join DG, and let it meet the circle again in P, and draw CN perpendicular to AB, meeting DG in Q, and LG in N.

Because of the parallels NC, LK,
it will be as LK : LD :: NC : NQ;
consequently LKD : LDK :: NCQ : NQC,
and dividendo DK² : LDK :: CQ² : NQC,
and permutando NQC : LDK :: CQ² : DK² :: GQ² : DG².
But, Prop. II. DP : DG :: GP : GQ,
therefore PDG : DG² :: PGQ : GQ²
but, Prop. XXI. PGQ = NQC; and PDG = ADB;
therefore ADB : DG² :: NQC : GQ²
permutando ADB : NQC :: DG² : GQ²;
invertendo NQC : ADB :: GQ² : DG².
Whence NQC : ADB :: NQC : LDK;
therefore the rect. ADB = the rect. LDK.

Q. E. D.

The

The same by Messrs. Harris, Lowry, and Swale.

ANALYSIS, by Messrs. Harris and Swale.

Draw HDM to meet the circle in M, and join GM.

By hypothesis the rect. LDK = the rect. ADB,
and, by Prop. XI. $ADB + DK^2 = GKH$;

therefore $LDK + DK^2$, ie, $LKD = GKH$;

therefore $LK : GK :: KH : KD$;

wherefore the Δ 's LGK, HDK, are equiangular

and therefore the $\angle HDK = \angle LGK = \angle GMH$;

Hence, GM is parallel to DE, that is, perpendicular to AB,
consequently by Prop. VII. $AC : CB :: AD : DB$.

Q. Q. V.

SYNTHESIS, by Mr. Lowry.

Since $AC : CB :: AD : DB$,
by conv. Prop. VII. GM is perpendicular to AB,
and therefore parallel to DE.

Hence, the $\angle HDK = \angle GMH = \angle LGK$;

therefore the Δ s LGK, HDK, are equiangular;

wherefore $KD : KH :: GK : LK$;

therefore LKD , ie, $LDK + DK^2 = GKH$;

but, Prop. XI. $ADB + DK^2 = GKH$;

therefore the rect. LKD = the rect. ADB.

Q. E. D.

The same by Mr. Nicholson.

Draw CF perpendicular to AB, and join LO,
FO, GO, DG, HO, and DF; O being the centre.

Then, by Prop. XV. DF is a tan. to the circle at F,

consequently $DO : FO = GO :: FO = GO : CO$;

theref. Eu. VI. 6, the $\angle GDO = \angle CGO = \angle OHG$;

therefore the points H, O, G, D, are in a circle,

the diameter of which is perpendicular to GH:

but, since OGL is a right angle, OL is the diameter

of the circle passing through H, O, G, D;

wheref. the $\angle COG = \angle DLG$, and $\angle CGO = \angle OLG$;

confe.

conseq. the $\angle DLO = COG + CGO (=DCK)$;
therefore, by sim. Δ s $LD : DO :: DC : DK$,

or, $LDK = ODC = DF^2 = ADB$.

Q. E. D.

PROP. XXIII.

Fig. 219, Pl. 15, and Fig. 224, Pl. 16.

*Demonstrated by Messrs. Campbell, Harris, Lowry,
Nicholson, and Swale.*

ANALYSIS, by Messrs. Harris and Swale

By hypothesis the rect. $ACB =$ rect. GCF ,
that is, $AC : FC :: GC : BC$,
and the angles GCB, ACF are right ones;
therefore the Δ s GBC, FCA , are equiangular,
and the $\angle CGB = \angle CAF$;
but, the $\angle GFE = \angle AFC$;
therefore $\angle GEF = \angle ACF =$ a right angle;
wherefore AB is the diameter of the circle.

Q. Q. V.

SYNTHESIS, by Messrs. Campbell, Lowry, and
Nicholson.

Since AB is the diameter of the circle,
the $\angle GEF$ is a right angle;
but, the $\angle ACF$ is a right angle;
therefore the $\angle GCB = \angle CAF$;
Hence, the Δ s GCB, ACF , are equiangular;
wherefore $AC : CF :: CG : CB$;
therefore the rect. $ACB =$ rect. GCF .

Q. E. D.

PROP. XXIV.

Fig. 220, Pl. 15, and Fig. 225, Pl. 16.

Demonstrated by Messrs. Campbell, Harris, Lowry, Nicholson and Swale.

ANALYSIS, by Messrs. Harris and Swale.

Draw GD, and let it meet the circle in I, and join HI. By hypothesis $LDM = DE^2 = DF^2$, that is, $DM^2 + LMD = DM^2 + FME$, or, $LMD = FME = GMN$; wherefore $LM : MN :: MG : MD$, therefore the Δ 's LNM, MGD, are equiangular; therefore $\angle NLM = MGD$ (NGI) = NHI; and theref. HI is parallel to DE, *ie.* perp. to AB; wheref. by Prop. VII. $AC : CB :: AD : DB$.

Q. Q. V.

SYNTHESIS.

By Messrs. Campbell, Lowry, and Nicholson.

Since $AC : CB :: AD : DB$, by conv. Prop. VII. HI is perpendicular to AB, and therefore parallel to DE; theref. $\angle MGD = NGI = NHI = NLM$; wherefore the Δ 's LNM, MGD, are equiangular; therefore $ML : MN :: MG : MD$; hence, $LMD = GMN = FME$, that is, $DM^2 + LMD = DM^2 + FME$, that is, $LDM = DF^2 = DE^2$.

Q. E. D.

PROP XXV. Fig. 226, 227, Pl. 16.

Demonstrated by Messrs. Campbell and Lowry.

Draw the diameter DM, and join FM, FE; and let CD meet GB in L.

Then

Then $\angle (DMF) GCL + (FDM) HFE = \text{a right } \angle$,
 and $\angle GCL + CGE = \text{a right } \angle$,
 therefore the $\angle CGE = \angle HFE$,
 and therefore the Δ 's GEF, HFE , are equiangular.
 Consequently $GE : EF (AE) :: (EF) AE : EH$;
 and therefore the rect. $GEH = AE^2$.

Q. E. D.

The same by Mr. Harris.

ANALYSIS.

Draw the diameter FI , and join IC .
 By hypothesis $GEH = AE^2 = EF^2$,
 that is, $GE : EF :: EF : EH$;
 therefore the Δ 's EHF, GFE , are equiangular.
 Hence the Δ 's HDL, FIC , are also equiangular;
 therefore $\angle HLD = GCI = \text{a right angle}$,
 and therefore CD is perpendicular to AB .

Q. Q. V.

The same by Mr. Swale.

ANALYSIS.

By hyp. $(AH + HE) \times AE = AE^2 = GEH = (GA + AE) \times HE$,
 hence, $HA \times AE = GA \times HE$;
 therefore $HA : HE :: GA : AE$,
 that is, $HA : 2HE :: GA : 2AE$,
 that is, $HA : HB :: GA : GB$;
 wheref. Conv. Prop. VII. CD is perpendicular to AB .

Q. Q. V.

PROP. XXVI. Fig. 228, 229, Pl. 16.

Join FD, DG ; let O be the centre, and draw FO, GO .
 Then (dem. Prop. IV.) $\angle FDC = CDG$; confe. $\angle KDF = GDL$.
 Also, by Prop. I. $DC \cdot CO = AC \cdot CB = FC \cdot CG$;
 therefore a circle will pass thro' the points F, D, G, O ;

G 2

wherefore

wherefore $\angle FDG + \angle FOG = 2 \text{ right } \angle's = \angle FDG + 2 \angle LDG$,
 and $\angle FOG = 2 \angle KHG$;
 therefore $\angle LDG = \angle KHG$,
 and therefore the $\Delta's$ LDG, LKH, are equianu
 consequently $LG : LD :: LK : LH$;
 therefore $LG \cdot LH = DL \cdot KL = KD \cdot DL + DL^2$;
 but, Prop. XI. $LG \cdot LH = DA \cdot DB + DL^2$,
 consequently the rect. KDL = rect. ADB.
Q. E. D.

The same by Messrs. Harris and Swale.

ANALYSIS.

Through O, the centre of the circle, draw the diameter GOI, and join HI, DG.

By hypothesis the rect. LDK = rect. ADB,
 and, by Prop. XI. $LD^2 + ADB = HL \cdot G$;
 therefore $(LD^2 + LDK)$ or, $KLD = HL \cdot G$;
 wherefore the points K, D, H, G, are in a circle;
 consequently $\angle GHK = \text{sup. } \angle GDK = \angle GDL$;
 therefore $\angle ODG = \angle FHI = \angle FGI = \angle CGO$;
 wherefore the $\Delta's$ OGC, ODG, are equiangular;
 therefore $DO : OG$ or $OA :: OG$ or $OA : OC$,
 and therefore $DO - OA : DO + OA :: OA - OC : OA + OC$,
 that is, $AD : DB :: AC : CB$.
Q. Q. V.

The same by Mr. Lowry.

Draw DF to meet the circle in Q, and join GQ.
 Then, since $AC : CB :: AD : DB$,
 by Prop. VII. QG is perpendicular to AB, and
 parallel to DE, therefore $\angle KLG = \angle QGH$;
 But the points FHQG are in a circle,
 therefore $\angle KFD = \angle QGH = \angle KLG$;
 and therefore the points D, L, H, F, are in a circle,
 Hence, DKL , or, $DK^2 + KDL = FKH$;
 but Prop. XI. $DK^2 + ADB = FKH$,
 and therefore the rect. KDL = rect. ADB.
Q. E. D.

PROP.

PROP. XXVII.

Fig. 230, 231, Pl. 16.

Demonstrated by Messrs. Campbell and Lowry.

Draw KC to meet the circle in M, and join HM.
 By hyp. $EC \cdot CF$, or, $EF \cdot FC + FC^2 = CD^2$;
 theref. $EFC = CD^2 - FC^2 = (CD + FC)(CD - FC) = NFD = KFG$;
 wherefore $EF : FK :: FG : FC$;
 therefore the Δ 's EFG, KFC, are equiangular;
 hence, $\angle KCE = FGE = HMK$;
 theref. HM is parallel to DN, and conseq. perp. to AB;
 consequently, by Prop. VII. $AL : LB :: AC : CB$.

Q. E. D.

The same by Messrs. Harris and Swale.

Let O be the centre of the circle; draw the diameter HOI, and join IK, HC.

By hypothesis $AL : LB :: AC : CB$,
 that is, $LO - AO : LO + AO :: AO - OC : AO + OC$;
 therefore $LO : OA$ or $OH :: OH : OC$;
 wherefore the Δ 's OIH, OCH, are equiangular;
 theref. the angles OHL, OCH are equal, and so are
 their complements, $\angle HCF = HIK = EGF$;
 wherefore the points H, C, G, F, are in a circle;
 consequently rect. $FEC = \text{rect. } HEG = \text{rect. } DEN$,
 or, $FEC + EC^2 = DEN + EC^2$,
 that is, $ECF = DC^2$.

Q. Q. F.

ARTICLE XIV.

*Three Propositions from Lawson.**(To be answered in Number IX.)*

PROP. XXXI.

LET AB touch a circle in B, and any line AE be drawn and therein be taken two points E and F on the same side of A, such that the rectangle EAF may be equal to the square of AB, and from A any line be drawn to meet the circle in C and D, and EC, FD be drawn meeting the circle again in G and H; GH being drawn will be parallel to AE.

PROP. XXXII.

Through any point A within a circle let a line be drawn meeting it in B and E, and therein two points F and G be taken such on different sides of A that the rectangle FAG may be equal to the rectangle BAE, and through A any line be drawn meeting the circle in C and D, and FC, GD be drawn to meet the circle again in H and K; then HK being drawn will be parallel to AB.

PROP. XXXIII.

Let AB be a line without a circle, and from A and B two lines be drawn to touch the circle in C and D, and let the square of AB be equal to the sum of the squares of AC and BD, and from A any line be drawn to meet the circle in E and F, and BE, BF be drawn meeting the circle again in G and H; the points A, G, H, will be in a right line.

ART.

ARTICLE XV.

*Two Propositions from Stewart's Theorems.**(To be answered in Number IX.)**PROP. XXVII. THEO. XXIV.*

LET there be any regular figure inscribed in a circle, and from all the angles of the figure and the centre of the circle let there be drawn right lines to any point; the sum of the fourth powers of the lines drawn from the angles of the figure, will be equal to the multiple by the number of the sides of the figure of the fourth power of the semi-diameter of the circle, together with four times the multiple by the same number of the fourth power of the line whose square is equal to the rectangle contained by the semi-diameter and the line drawn from the centre, together with the multiple by the same number of the fourth power of the line drawn from the centre.

PROP. XXVIII. THEO. XXV.

Let there be any regular figure of a greater number of sides than four circumscribed about a circle, and from any point in the circumference of the circle let there be drawn perpendiculars to the sides of the figure; eight times the sum of the fourth powers of the perpendiculars, will be equal to 35 times the multiple by the number of the sides of the figure of the fourth power of the semi-diameter of the circle.

ART.

ARTICLE XVI.

On the Description of Parabolic Trajectories.

By Mr. WILLIAM WALLACE, *Assistant Teacher of the Mathematics in the Academy at Perth.*

PROP. I. THEOREM. Fig. 217, Pl. 15.

IF three straight lines touch a parabola, a circle described through their intersections shall pass through the focus of the parabola.

Let DC, DE, CE be three tangents to a parabola, a circle described about the triangle DCE shall pass through F the focus of the parabola.

This Theorem was proposed as a prize Question, at page 309, vol. I. and the demonstration thereof together with the demonstrations of the following corollaries are given at pages 56 and 57, Vol. II.

Cor. 1. If CD, CE touch the curve at A and B, and if FA, FB be joined, the triangles AFC, CFB are similar to the triangle DFE.

Cor. 2. Let AC, BC be two straight lines given by position touching a given parabola, if any other tangent be drawn meeting them in D and E, and FD, FE be joined, the triangle DFE is given species.

Cor. 3. If AC, BC are two tangents to a parabola, and if a circle be described through the focus to touch BC, one of them, at C, it shall pass through A the point where the other tangent meets the curve.

Cor. 4. If AC, BC, DE are tangents to a parabola at the points A, B, N; these lines are similarly divided at their intersections and the points of contact, that is $AD : DC = DN : NE = CE : EB$.

The

The following problem is very simple, and only given for the sake of those that follow.

PROP. II. PROBLEM. Fig. 252, Pl. 16.

To describe a parabola, having given its focus and two tangents to the curve.

Let H be the given focus, and PB, BQ the two given tangents. Draw HG, HK perpendicular to the tangents, and in the perpendiculars produced take GM equal to GH and KR equal to KH; join MR which will evidently be the directrix, and the focus being given, the curve may be described in the manner taught by the writers on Conic Sections.

PROP. III. PROBLEM. Fig. 252, Pl. 16.

To describe a parabola that may touch four straight lines given by position.

Let AB, AC, BF, DF be the four straight lines, and let ABC, DBF be any two of the four triangles formed by the lines; let circles be described about these triangles, they will meet each other at H, the focus of the parabola. Having now given the focus and more than one tangent to the curve, the manner of describing the parabola is evident from the last problem.

PROP. IV. PROBLEM. Fig. 252, Pl. 16.

Three tangents to a parabola, and the point at which one of them touches the curve, are given, to describe the parabola.

Let AB, BC, AC be the three straight lines given by position, and let P be the given point of contact. Describe a circle about the triangle ABC, also through P describe a circle that may touch BC,
any

any one of the lines, at B; this circle will cut the other at H the focus of the parabola; having now the focus and more than one tangent to the curve the method of describing it is obvious.

PROP. V. PROBLEM. *Fig. 252, Pl. 16.*

Two tangents to a parabola, and the points at which they touch the curve are given to describe the parabola.

Let BP, BQ be the tangents, and P, Q the points of contact. Through the point P describe a circle that may touch the tangent BQ at B; also through the point Q describe a circle to touch BP at B; these circles shall meet each other at H the focus of the parabola required. There are now given two tangents and the focus, hence the method of describing the curve is evident.

PROP. VI. PROBLEM. *Fig. 252, Pl. 16.*

To describe a parabola that shall touch three given straight lines and have its axis parallel to a straight line given by position.

Let AB, BC, AC, be the three straight lines given by position, and let de be the straight line to which the axis is to be parallel.

Describe a circle about the triangle ABC, let de meet the tangent AB in d , make the angle CBH equal to the angle edB and H will be the focus of the parabola required.

Let the curve be described, and let it touch BA at P, join HP. By Prop. I. the angle HBC is equal to the angle HPB, but by construction HBC is equal to the angle edB , and it is evident that HPB is equal to the angle contained by the tangent AB and the axis, hence the axis and de make equal angles with AB, therefore the axis is parallel to de as was required.

ART.

ARTICLE XVII.

Answers to the Mathematical Questions proposed in ARTICLE LVIII. No. V. Vol. I.

I. QUESTION, 89, answered by Mr. J. Lochwood.

AS the side of the equilateral triangle inscribed in the generating circle is given $= 16$, the diameter of the circle is readily found $= \frac{8^2}{3} \sqrt{3} =$ the axis of the cycloid; therefore when the cycloid revolves about a tangent parallel to its axis, the diameter of its circumscribing cylinder will be expressed by $\frac{64}{3} \sqrt{3} \times 3 \cdot 1416 = 64 \times 1 \cdot 0472 \sqrt{3}$. But by what is said under the article *cycloid* in Dr. Hutton's Dictionary, this content is to the content of the solid generated by a revolution of the cycloid about a tangent parallel to its axis, as 4 : 3; therefore, $3072 \times (1 \cdot 0472)^3 \times \cdot 7854 \times 32 \sqrt{3} = 146649 \cdot 90222$, the content required.

The same answered by Mr. Ralph Simpson, Croxdale, near Durham.

It has been shewn that the side of an equilateral triangle is to the radius of its circumscribing circle as 1 to 57735027. Therefore as $1 : 57735027 :: 16 : 9 \cdot 23760432 =$ radius of the generating circle, which $\times 2 \times 3 \cdot 1416 = 58 \cdot 041715 =$ its circumference; which, according to the property of the cycloid, is $=$ the length of its base; and its height or axis $= 18 \cdot 4752 =$ the diameter of the circle.

According to Dr. Hutton's Mathematical and Philosophical Dictionary, Vol. I. pa. 355, Torricelli has shewn that the content of a solid, formed by the rotation of a cycloid about a tangent parallel to its axis, is to the content of its circumscribing cylinder as 3 to 4. Now the content of a cylinder whose altitude is $18 \cdot 4752 =$ the axis of the cycloid, and diameter of the end $116 \cdot 08343 =$ twice the cycloid's base, is easily found $= 195533 \cdot 15315$. And therefore, by the above proportion, $195533 \cdot 15315 \times 3 \div 4 = 146649 \cdot 86486$ the content required.

Ingenuous solutions were also given by Messrs. Bolworth, Gregory, Harris, Johnson, Lowry, and Swale.

II. QUES-

II. QUESTION 90, answered by Mr. Johnston, Birmingham.

Conf. Let ABHI (fig. 232, pl. 16,) be the rectangular garden, and on AB produced take $BD = AH$; draw DK perp. to AD, and $= \frac{1}{20}$ th of AB; join HK, upon which as a diameter let a circle be described intersecting AD in O, and AO will be the breadth of the walk.

Demon. Let S be the centre of the circle and draw SV parallel to AH meeting AD in V; also let L be the point where the circle cuts AH, and join OS, LS, and KL; on BI take $BM = AO$; then it is evident that the area of the walk is $= BM \times (AB + MI) = BM \times OD$; but by the property of the circle $AO = DQ$, and $AL = DK = AB \div 20$; therefore $AO \times OD (= BM \times OD) = OA \times AQ = LA \times AL = \frac{1}{20} AB \times AH$; that is, the area

of the walk is $=$ to $\frac{1}{20}$ th of the area of the garden.

Calculation, $DK = 4$, $LK = AD = 180$, $LH = 96$, $SG = 48$, $SV = 52$, $OS^2 = LS^2 = 90^2 + 48^2$, and $OV = \sqrt{(OS^2 - SV^2)} = \sqrt{(90^2 + 48^2 - 52^2)} = \sqrt{7700} = 87.749643874$; hence, $AO = 2.250356126$ yards the breadth of the walk required.

The same by Mr. Richard Nicholson, Liverpool.

Conf. Fig. 233, pl. 16. Take AB, BC equal to the length and breadth of the garden respectively; bisect AC in O, and take AF to FB as the area of the walk to the area of the garden. Upon FC describe a semi-circle to meet BD, a perp. to FC, in D; then with the centre O and distance OD let the semi-circle GDH be described, then AG or HC will be the breadth of the walk required.

Demon. $AB \cdot BC - GB \cdot BH (FB \cdot BC) = AF \cdot BC : AB \cdot BC :: AF : AB$, the given ratio, as per *Conf.*

Cal. BF is $= 95$, and $BC = 80$; hence $BD^2 = 95^2 + 80^2 = 7600$; but OB is $\frac{180}{2} - 80 = 10$; therefore $OD = \sqrt{(BD^2 - OB^2)} = \sqrt{7700} = 87\frac{1}{2}$, and $HC = OC - OD = 2\frac{1}{2}$ yards nearly.

The

The same algebraically by Mr. W. Marrat, Lincoln.

Put x for the breadth of the walk; then $100 - x =$ length of the middle walk, and $x(100 - x) =$ its area; also $80x$ is the area of the end walk; and their sum $x(100 - x) + 80x = 8000 \div 20$, that is, $180x - x^2 = 400$; hence $x = 2\frac{1}{2}$ yards nearly.

The same otherwise by Mr. Gregory.

This Question may be readily solved by a quadratic equation, and it will admit of a neat geometrical construction; but I choose rather to answer it by the common rule of trial and error, thus: Let it be supposed that 2 is the breadth of the walk, then is $100 - 2 = 98$, the length of the walk along the middle; hence, as 80 is the breadth of the Garden, $2 \times 80 + 98 = 356$, for the content of the whole walk, instead of $\frac{100 \times 80}{20}$ or 400; an error of 44

in defect. Again, suppose $2\frac{1}{2}$ the breadth; then $97\frac{1}{2}$ the length of the walk along the Middle; therefore $97\frac{1}{2} + 80 \times 2\frac{1}{2} = 443\frac{1}{2}$, instead of 400, an error of 43 $\frac{1}{2}$ in excess. The sum of the products of the first assumption into the second error, and the second assumption into the first error, is 197 $\frac{1}{2}$; which divided by 87 $\frac{1}{2}$, the sum of the errors, gives $2\frac{88}{351}$ or nearly $2\frac{1}{2}$ for the width

of the walk. If this be assumed for the width, the operation pursued as before, and the result corrected by either of the former assumptions and its result, we shall get 2.2503562 for the answer; agreeing with $90 - 10\sqrt{77}$, the answer obtained by a quadratic equation, to the 7th decimal place.

Other ingenious answers were also received from Messrs. Bosworth, Harris, Johnlston, Lockwood, Lowry, Reed, Swale, and Thornoby.

III. QUESTION 91, answered by Mr. Lowry.

Let QST (Fig. 234, pl. 16.) be a section of the conical glass, BPAR a section of the spheroidal egg passing through the centre C. Here CD the semi-conjugate $= \frac{1}{2}$, BA the tranverse $= 2$, TE $= 4$, QS $= 2$, and ES $= 1$. Then by Emerson's Conics 1. 26, $AK \times BH = CD^2$: but $AT = 4 AK$ and $TB = 4 BH$; therefore $AT \times TB = 16AK \times BH = 16CD^2$.

VOL. II.

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On AB describe a semi-circle and draw the tangent FG ; join CG, then $TG^2 = AT \times TB = 16CD^2$; therefore $TC^2 = CG^2 + 16CD^2$, or $TC = \sqrt{5}$; hence EC and consequently AE, BE are given ; therefore, by the property of the ellipsis we have $BA^2 : FD^2 :: AE \times EB : NE^2$; therefore NE is given. Hence the content of the immersed part of the egg may be found, by Mensuration, from which deducting the content of the vacuity of the glass, (which may easily be found,) the remainder will be the quantity of water overflowing. But the numbers given in the question are such as shew the content of the vacuity of the glass to be greater than the content of the whole egg ; consequently no water will run over in the present instance.

The same by Mr. J. Lockwood, Leeds.

Let STQ (fig. 234, pl. 16.) represent the glass, APBR the spheroid, and put $BA = a = t$; conjugate axe $= c$; and $AM = x$; then from known principles the subtangent TM will be expressed by $(2tx - 2x^2) \div (t - 2x)$, and the ordinate PM by $(c \div t) \times \sqrt{(tx - x^2)}$: but by sim. Δs TM : MP :: 4 : 1, per question ; therefore $(2tx - 2x^2) \div (t - 2x) = (4c^2 \div t) \times \sqrt{(tx - x^2)}$, and by reduction $(t^2 + 16) x^2 - (16t + t^2) x + 4c^2 = 0$, or in numbers $x^2 - 2x + \frac{4}{5} = 0$; hence $x = 1 - \sqrt{(1 \div 5)}$. Now it will be very easy to determine TM, PM, TB, BE, AE and consequently NE. And then every thing else, if proper numbers had been given by the proposer.

The same by Mr. T. Reed, Student in Mr. Bulmer's School at Sunderland, Durham.

The solidity of the egg is $17 \times 2 \times .5236 = 1.0472$; content of the whole cone $= 27 \times .7854 \times (4 \div 9) = 4.1888$; content of the water contained in it is $= 1.757 \times .7854 \times 8.5 \div 3 = 3.80616876$; therefore $4.1888 - 3.8061 = 1.3826 =$ the content of the vacuity of the glass, and as this exceeds the content of the egg, no water will run over.

And thus the question is answered by Messrs. Bosworth, Gregory, Harris, Lowry, Marrat, Simpson, Swale, and Thornoby.

VI. QUESTION 92, answered by Mr. Lockwood.

Let ABCD (fig. 235, pl. 16.) represent the semi-circular field ; P the given point in the diameter, and draw the lines as in the figure.
Put

Put $AB = 2a$, $OP = b$ and $AE = x$; then by the property of the circle $AD = \sqrt{2ax}$, $ED = \sqrt{(2ax - x^2)}$: also the area of the sector AOD, or BOC will be expressed by $(\frac{1}{2} \sqrt{2ax} - \frac{1}{2} \sqrt{(2ax - x^2)}) \times (a \div 6)$; and the area of the Δ OPD, or OPC by $b \sqrt{(2ax - x^2)}$; therefore the area of the space PAD, or PBC will be expressed by $(\frac{1}{2} \sqrt{2ax} - \frac{1}{2} \sqrt{(2ax - x^2)}) \times (a \div 6) + \frac{1}{2} b \sqrt{(2ax - x^2)} = 200 \div 3$ by the question; which equation gives $x = 2.68$, or 11.6046 . Hence $PD = 16.327$, and $PC = 12.628$.

Also by Trigonometry, the \angle APD is found $= 26^\circ 33' 39''$, and the \angle APC $= 63^\circ 16' 36''$.

Whence the expence of fencing will be 15l. 18s. 6d.

In the same way it was answered by Messrs. Bosworth, Gregory, Johnston, Lowry, Swale, and Thornoby.

V. QUESTION 93, answered by Mr. Samuel Thornoby.

Since this question must be solved by some approximation or other, perhaps the following is as ready as any. Since the areas of similar segments of circles are as the squares of their versed sines, the easiest method of finding the diameter, will be by first finding a similar segment in a table of areas and versed sines.

This may easily be done by means of the large Table at the end of Dr. Hutton's Mensuration, and by a few trials, the versed sine of a segment similar to that in the question is .031, its area being $= .00720942$; for $.00720942 \div (.031)^2$ is $= 7.508$, and $30 \div 2 = 7.5$, which quotients are sufficiently near an equality. Hence, then as $.031 : 2 :: 1$ (tab. diam.) : $64\frac{1}{2}$ very near the true diameter, and therefore the chord of the segment is $= 22.36$ chains.

Now, from the nature of the question the space GPCBG (fig. 236, pl. 16.) must be to the space AIHFA as 40 to 30, or as 4 to 3. Assume $CL = .4$ chains; then the area of the segment $PCL = 2.701628$, and the area of the rectangle $GPLD = 8.1016$; hence, 4 times the space $GPCBG = 103.212912$, and 3 times the space $AIHFA = 99.6444$; therefore the error is 3.567512 in excess.

Again, suppose $CL = .3$ chains, then proceeding after the same manner, the error will be 2.504 too little. Hence we find

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On AB describe a semi-circle \equiv
 join CG, then $TG^2 = AT \times TB =$
 therefore $TC^2 = CG^2 + 16 CD^2$, or
 EC and consequently AE, BE are
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 therefore NE is given. Hence the
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 the content of the vacuity of the glass, (which
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 numbers given in the question are such
 vacuity of the glass to be greater than the
 consequently no water will run over in the

The same by Mr. J. Locke

Let STQ (fig. 234, pl. 16.) represent
 spheroid, and put $BA = 2 = r$; conjugate
 x ; then from known principles the substance
 by $(2x^2 - x^2) \div (2 - 2x)$, and the ordinates
 by $(2x^2 - x^2) \div (2 - 2x)$. $\Delta^2 TM = 3x^2$
 $(2x^2 - x^2) \div (2 - 2x)$ but by sim. $\Delta^2 TM = 3x^2$
 therefore $(2x^2 - x^2) \div (2 - 2x) = (16t + 16)$
 by resolution $(x^2 + 16) x^2 = 0$; hence $x =$
 4 or $x^2 - 2x + 1 = 0$; to determine T the
 it will be very easy to determine T the
 consequently NK. And then every thing
 had been given by the proposer.

The due to Mr. T. Reed, Student of
Oxford, 1711

The Quantity of the egg is 7
 and the whole cone is 27
 and the water contained is 20
 and the glass is 7
 and the glass is 7

$+ (4 \div 15) \times \sqrt{2mx}) = \text{area of the curvilinear space AFPHI}$
 Whence by the nature of the question $(v - u) ((3 \div 5) (x - a) \sqrt{(2mx - x^2)} - (4 \div 15) \times \sqrt{2mx + n}) + u (c - \sqrt{(2mx - x^2)})^2 = 2uv$. From which equation we obtain $x = 3904285$ chains; consequently $DG = 5.002834$, $GA = 6.177166$ chains; also the area of the square $= 38.1565644$, the area of the part $AFPG = 6.061009$, and the area of the external part $AFPHIA = 32.095555$ square chains, all nearly as required.

According to one or other of these methods is the answers given by Messrs. Bosworth, Gregory, Johnston, Lowry, Simpson, and Thornoby.

VI. QUESTION 94, answered by Mr. Johnston.

Let HR (fig. 183, pl. 13.) be the horizon, P the pole, Z the zenith, and A the sun's place at the time of observation. Then we have given the colat. $PZ = 38^\circ 28'$, the sum of the coalt. and codec. $AP + AZ = 110^\circ 12'$, the sum of the azimuth and hour angle $AZP + APZ = 156^\circ 35'$, to find the angle APZ.

Now by Emerson's Trigonometry, B. III. Prop. 37.

$$\begin{aligned} \text{As tangent } \frac{1}{2}PZ &= 19^\circ 14' \\ \therefore \text{tangent } \frac{1}{2}(AP + AZ) &= 55^\circ 6' \\ \therefore \text{cotine } \frac{1}{2}(AZP + APZ) &= 78^\circ 26' 30'' \\ \therefore \text{cotine } \frac{1}{2}(AZP - APZ) &= 34^\circ 36' 30''. \end{aligned}$$

Hence the azimuth is $= 113^\circ 3'$, and the hour angle $= 43^\circ 50'$; consequently the time from noon is 2 hours. 55 min. 20 sec.

And thus the question is answered by Messrs. Bosworth, Gregory, Harris, Lowry, Simpson, Swale, and Thornoby.

VII. QUESTION 95, answered by Mr. Louis Hill.

Let ACBD (fig. 238, pl. 16.) be the given ellipse, AB, CD its transverse and conjugate axes respectively; F, G its foci. Suppose the point A to have arrived at any other point Q in the quadrant AQC; join QF, QG, and in QF produced take $FP = QG$, and P will be a point in the curve required. Let PE, QI be made perp. to AB and put $AB = t$, $FG = a$, $FB = d$, the abscissa $BE = x$ and the corresponding ordinate $PE = y$. Then $EF = d - x$, $FP = \sqrt{y^2 + (d - x)^2}$ and $FQ = t - \sqrt{y^2 + (d - x)^2}$; also, by sim. Δ s $FP : FQ :: FE : FI = t - \sqrt{y^2 + (d - x)^2} \times (d - x) \div \sqrt{y^2 + (d - x)^2}$. By the question and the property of the figure $QG + QF = PF + QF = AB$,

= AB, and by Simpson's Geometry B. II. Theo. IX. $FP - QF = GQ - QF = GI - FI$, that is, $FP - QF : FG :: GI + IF = FG + 2IF : AB$; but $FP - QF$ is = $2FP - AB$

= $2\sqrt{y^2 + (d-x)^2} - t$, therefore $2\sqrt{y^2 + (d-x)^2} - t : a :: a : + (2t - 2\sqrt{y^2 + (d-x)^2}) \times (d-x) \div \sqrt{y^2 + (d-x)^2} : t$, or $2t\sqrt{y^2 + (d-x)^2} - t^2 = a^2 + (2at - 2a\sqrt{y^2 + (d-x)^2}) \times (d-x) \div \sqrt{y^2 + (d-x)^2}$; this by reduction gives $2t(y^2 + (d-x)^2) - 2at(d-x) = (t^2 + a^2 - 2a \times d - x)\sqrt{y^2 + (d-x)^2}$ for the equation of the curve required, which is a line of the 4th order.

Obs. When x is = 0, y is = 0, and when $x = d$, y is = $(t^2 + a^2) \div 2t = FH = FR$ the greatest ordinate.

Similar answers were also received from Messrs. Bosworth, Gregory, Evans, Lowry, Surtees, and Swale.

VIII. QUESTION 96, answered by Mr. Olinthus Gregory, Bookseller and Teacher of the Mathematics, Cambridge.

Let AB (fig. 239, pl. 16.) represent the vertical side of the vessel, a, c, e, g , &c. the several apertures therein, at a foot distance from each other: on AB describe a semicircle, and let the lines ab, cd, ef, hg , &c. be drawn, then (as it is shewn by most writers on Hydrostatics) twice ab , is the distance which water spouting out at a falls from B, on the horizontal BH, twice cd , is the distance which water spouting out at c falls from B, and so on. But it is a property of the circle that $Aa \times aB = ab^2$, $Ac \times cB = cd^2$, &c. hence the several distances to which the fluid will spout are readily found to be as follows:

$$2\sqrt{1 \times 9} = 2\sqrt{9} = 6.$$

$$2\sqrt{2 \times 8} = 2\sqrt{16} = 8.$$

$$2\sqrt{3 \times 7} = 2\sqrt{21} = 9.16515$$

$$2\sqrt{4 \times 6} = 2\sqrt{24} = 9.79795$$

$$2\sqrt{5 \times 5} = 2\sqrt{25} = 10.$$

$$2\sqrt{6 \times 4} = 2\sqrt{24} = 9.79795$$

$$2\sqrt{7 \times 3} = 2\sqrt{21} = 9.16515$$

$$2\sqrt{8 \times 2} = 2\sqrt{16} = 8.$$

$$2\sqrt{9 \times 1} = 2\sqrt{9} = 6.$$

In order to find the quantity of water discharged at all the holes, first take that at 1 foot deep; put $s = 16 \frac{1}{12}$ feet, $D = 1$, depth from the surface of the water to the centre of the hole, $F = .000218166$ feet, area of the hole, $t = 600''$ time of discharging: then (Cor. 6. Prop. 97. *Emerson's larger Mechanics*) $6.128tF\sqrt{2Ds} = 4.54946$ ale gallons, discharged by the upper hole in 10 minutes. And as it is known that the quantities discharged from equal holes in equal times are as the square roots of their depths below the surface; 4.54946 the quantity discharged by the upper hole, being multiplied into the square roots of 2, 3, 4, 5, 6, 7, 8, and 9, respectively, will give 6.43391, 7.87990, 9.09892, 10.17290, 11.14386, 12.03674, 12.86782, and 13.64838 for the several quantities issuing at each of the holes; the sum of which is 87.83189 ale gallons, discharged by all the 9 holes in 10 minutes.

Thus is the quantity of water discharged determined on the supposition that the velocity with which it spouts is equal to that which a heavy body would acquire in falling through half the height of the surface of the water above the aperture. But neither this hypothesis, nor any other which has yet been advanced, agrees with experiments, when applied to ascertaining the measure of effluent water: indeed scarcely any of the attempts which have been made to prove the truth of the two most generally received hypotheses [one of which is laid down by NEWTON in the first edition of his *Principia*, and the other in the second and third editions] are satisfactory; and, if the modes of reasoning which have been used are suspicious, but little dependance can be placed on the results.

Perhaps the best information on this subject may be met with in the works of *M. D'Alembert* and *D. Bernoulli*, and in an ingenious paper, by the *Rev. M. Young*, D. D. inserted in the *Transactions of the Royal Irish Academy* for 1788.

N. B. Since the above was written I have perused the second volume of *Dr. Hutton's* useful "Course of Mathematics", and there find an answer to this question, in which the *distances* agree with those in this solution, and the *quantity* discharged is determined to be 123.8849 gallons, on the supposition that the velocity of water is equal to that acquired by a heavy body in falling through the *whole* height of the water above the orifice. It may be worth while to observe that if the quantity discharged, as determined in my solution be multiplied into $\sqrt{2}$, the product will very nearly agree with the quantity given in *Dr. Hutton's* answer.

Ingenious solutions to this question were also received from Messrs. Bosworth, Lowry, Marrat, and Simpson, the proposer.

To IX. QUESTION 97, *no answer has been received.*

X. QUESTION 98, *answered by Mr. Newton Bosworth, Peterborough.*

The first person who appears to have treated the subject of this question with any degree of accuracy, was the celebrated *Leibnitz*, in the *Acta Eruditorum* of Germany, anno 1685. His method of determining the absolute and relative pressure of the Globe on each of the planes, though scientific and ingenious, is yet, I presume, diffusive and inelegant. His paper, however, is a valuable one, and worth consulting by those who desire information on the subject. An English translation of it may be found in "*Acta Germanica*," or Literary Memoirs of Germany and the North, VOL. II. pa. 73.

This Problem has also been investigated by the late Mr. DITTON, in a publication entitled "*The General Laws of Nature and Motion.*" As his solution is more concise than that of Leibnitz, though evidently founded upon similar principles, I shall take the liberty of extracting it.

"Let the globe H rest upon the two inclined planes (fig. 240, pl. 16.) AC and CD; and let the angle ACD be a right one. Let G express the absolute gravity, or total *momentum* of the globe, which is shared betwixt the two planes (each of which bears its part) but in what proportion we are now to determine. Let $AC = L$, $AB = A$, $DC = l$, and $DE = a$. The angle ACD made by the two planes, being a right one by the hypothesis, 'tis evident, that the quadrilateral figure GHFC is a perfect square; or, which is all one, that the lines IHF, KGH, which are the lines of direction of the globe's perpendicular pressure upon each plane, are parallel to those planes respectively; viz. IHF parallel to AC, and KHG parallel to DC. And that this could not be were the angle ACD an acute or an obtuse one, is obvious to any one that considers the matter. From hence then it follows, that each plane performs to the other, the office of a power acting with a direction parallel to the plane. Thus the plane AC sustains all that force of the globe, by which it endeavours to descend upon the inclined plane DC, and which force, if the plane AC were taken away, or powers (acting with the direction KG parallel to the plane DC) would sustain. So likewise the plane DC, sustains the *momentum* of the globe upon the plane AC, which *momentum* would be balanced by a power acting with the direction IF parallel to AC, if the plane DC were removed. But besides these pressures, each plane is burthened with one of
another

another kind, and that is the difference between the total absolute *momentum* of the globe and that relative or partial *momentum* with which the globe endeavours to descend upon that plane. For, if either of the planes were taken away, and a power was substituted in the room of it, then though that power sustained the burden which pressed upon the plane before, yet the body still resting upon the other plane, that plane is pressed by it; and the quantity of that *momentum* by which the plane is pressed is evidently what was just now asserted. To state these proportions therefore, 'tis certain that the *momentum* by which the globe endeavours to descend upon the plane DC, is $= \frac{G \times a}{l}$; this the plane AC sustains. And the *momentum* by which the globe endeavours to descend upon the plane AC, is $= \frac{G \times A}{L}$; this the plane DC sustains. But the plane DC sustains also the *momentum* $G - \frac{G \times a}{l}$; and the plane AC sustains also the *momentum* $G - \frac{G \times A}{L}$. Therefore the whole *momentum*, with which the plane DC is burdened, is $= \frac{G \times A}{L} + \frac{G \times l - G \times a}{l}$; and the whole *momentum* with which the plane AC is burdened, is $= \frac{G \times a}{l} + \frac{G \times L - G \times A}{L}$. And the sum of these two aggregates of *momenta* (with which both planes are burdened) casting out contradictions, will be $= \frac{G \times l \times L + G \times l \times L}{L \times l} = 2G =$ twice the absolute *momentum* of the globe; which is absurd and impossible. For the absolute *momentum* of the globe is parted between the two planes, and what they sustain cannot possibly exceed that. We are therefore to take the halves of these sums of *momenta*, or (which amounts to the same) an arithmetical mean betwixt them. And then they will be $\frac{G \times A}{2L} + \frac{G \times l - G \times a}{2l}$, and $\frac{G \times a}{2l} + \frac{G \times L - G \times A}{2L}$; the sum of which is $= G$,
the

the absolute *momentum* of the globe. The *momentum* of the globe then, with respect to each plane, is determined. Q. E. F."

But I apprehend the question will admit of a solution, much more elegant and simple, and yet equally decisive: this I have attempted to make out as follows.—

Since two sides of each of the right angled triangles ABC, CED, (fig. 240, pl. 16.) are given, the other side, being easily found, may be said to be given also.

In order to attain its present situation, suppose the globe to have rolled along the plane CD from D towards C, and that when arrived at F, it is stopped in its progress by the other plane AC, at G. Now it is evident that the pressure upon the plane AC, is equal in effect to a power capable of sustaining the globe upon the other plane, acting in the direction GK, parallel to CD. By Mechanics, as the power at K is to pressure on the plane CD, so is DE to CE. Therefore the pressure of the globe on the plane CD, is to that on the plane AC, as CE is to DE.

Again: Imagine the globe to have descended down the plane AC, and that it is stopped and sustained by the other plane CD, which bears a weight equal to a power acting at I, in the direction IF, and capable of holding up the globe if that plane were taken away. Then the pressure on AC is to that on DC, as BC is to AB, or, the pressure on DC is to that on AC, as AB is to BC: which is the same as before, for it may easily be proved that CE: DE :: AB: BC, and the sum of the pressures on both planes, as determined by each of the above methods, is equal to the whole weight, or total *momentum*, of the globe.

From hence, and the nature of the question, it follows, that, if the planes be of equal length, the quantity of pressure each sustains will be inversely as the altitudes; if the altitudes be alike, the pressure will be directly as their lengths; and if both planes be of equal length and altitude, the pressure will be equal also.

Thus the particulars of this interesting question are truly determined, without the "alternatives" of LEIBNITZ, or the "contradictories" of DITTON; and I presume the superior simplicity of the method I have used, will justify my deviation from authorities so respectable.

Ingenious solutions to this question were also received from Messrs. Lowry, Swale, and Thornoby.

XI. QUESTION 99, answered by Mr. W. Marrat, Lincoln

Let ABC (fig. 241, pl. 16.) be the horizontal plane, Aa, Bb, Cc the posts, and bD, cD, aD the rasters, also let bf, ae, cg, be

be perp. to DH. Draw cs , cn parallel to BC, CA. Then, s being a right angle, in the $\triangle bcs$ we have given the sides bs , cs and the angle s to find the side bc and the $\angle scb$; also in the $\triangle bcD$ we have the three sides given to find the angles Dbc , and Dcb . And the $\angle Dbf = \angle Dbc - \angle scb$; the $\angle Dcg = \angle Dcb - \angle scb$. Hence, in the $\triangle Dbf$ we have given Db and all the $\angle s$ to find bf , the distance of the post Bb from the point H, which is perp. under D. And in the $\triangle Dcg$ we have Dc and all the $\angle s$ to find cg the distance of the post Cc from the point H. In the same manner may the angles be found in the $\triangle Dae$, and then the side ae , which is the distance of the post Aa , from the point H.

The same answered by Mr. John Harris.

Let H, (fig. 241, pl. 16.) be the point sought, and join AH, BH, CH; also join DH and on it let fall the $\perp s$ ae , be , ce from the top of the posts Aa , Bb , Cc . Now, in the $\triangle ACB$, all the sides being given, the sine and cosine of the $\angle CAB$ is easily found, which put $= s$ and c respectively; also put the cosine of

the $\angle CAH = y$; its sine will be $= \sqrt{1 - y^2}$ (the radius being 1.)

and the cosine of the $\angle BAH$ will be $= cy + s \sqrt{1 - y^2}$.

Hence by a well known theorem, we shall have

$$AC^2 + AH^2 - 2AC \times AH \times y = CH^2,$$

$$\text{and } AB^2 + AH^2 - 2AB \times AH \times (cy + s \sqrt{1 - y^2}) = BH^2.$$

$$\text{But, Eu. I. 47, } CH = eg = cD - Dg = cD - (\sqrt{aD^2 - AH^2} + eg) = cD - aD - eg + AH^2$$

$$= 2eg \times \sqrt{aD^2 - AH^2} = (\text{because } eg = \text{post } Aa - \text{post } Cc = 1) cD - aD - eg + AH^2 = 2\sqrt{aD^2 - AH^2}.$$

$$\text{Theref. } AC^2 + AH^2 - 2AC \times AH \times y = cD^2 - aD^2 - eg^2 + AH^2 - 2\sqrt{aD^2 - AH^2}, \text{ or } 2AC \times AH \times y = 2\sqrt{aD^2 - AH^2}$$

$$+ aD^2 + eg^2 + AC^2 - cD^2, \text{ that is, } 2bxy = 2\sqrt{a^2 - x^2} + aD^2 + eg^2 + AC^2 - cD^2 = 2m \text{ and } AH = x;$$

$$\text{consequently } y = (\sqrt{a^2 - x^2} + m) \div bx$$

Again,

Again, $BH^2 = bf^2 = bD^2 - (\sqrt{aD^2 - AH^2} + ef)^2 =$ (because $ef = \text{poft } Aa - \text{poft } Bb = \frac{1}{2} bD^2 - aD^2 - ef^2 + AH^2 - \sqrt{aD^2 - AH^2}$; therefore $AB^2 + AH^2 - 2AB \times AH \times (cy + s \sqrt{1-y^2}) = bD^2 - aD^2 - ef^2 + AH^2 - \sqrt{aD^2 - AH^2}$, or $2AB \times AH \times (cy + s \sqrt{1-y^2}) = \sqrt{aD^2 - AH^2} + aD^2 + ef^2 + AB^2 - bD^2$, that is, $cx \times (cy + s \sqrt{1+y^2}) = \sqrt{a^2 - x^2} + n$ (putting $2AB = c$, and $aD^2 + ef^2 + AB^2 - bD^2 = n$). Whence, by substituting $(\sqrt{a^2 - x^2} + m) \div bx$ for y in this equation and reducing it, we get

$$x = \sqrt{\left(\frac{2qr - p^2}{2q^2}\right) + \sqrt{\left(\frac{2qr - p^2}{2q^2}\right)^2 - \frac{r^2 - p^2 a^2}{q^2}}}$$

Where $p = (2nb - 2mec)(b - ec) + 2me^2s^2$, $q = (b^2 + 1) \times e^2s^2 + (b - ec)^2$ and $r = (a^2 + m^2)e^2s^2 + (nb - mec)^2 + (b - ec)^2 \times a^2$.

XII. QUESTION 100, answered by Mr. Lowry.

Let AB (fig. 242, pl. 16.) represent the horizon, Z the zenith, P the pole; OP, PS, the hour circles making given angles ZPS, ZPO, with the meridian AZB; SZ, OZ, the azimuth circles, *mn* the parallel of declination, O and S the star's position, and OP = SP the co-declination required.

Now while the star passes over the portion OS of the parallel of declination, the increase of azimuth (not the increase of declination) must be the greatest possible; therefore the angle OZS must be a *maximum*, while the side ZP and angles ZPS, ZPO are constant, and the other sides variable. When this takes place it is evident, from *Cotes De Estima Error in mixt Mathem. Prop. XVI. that*

fin. ZO : fin. POZ :: fin. ZS : fin. PSZ, and by trigo.
fin. ZO : fin. ZPO :: fin. ZP : fin. ZOP; therefore
fin.² ZO : fin. ZPO :: fin. ZS \times fin. ZP : fin. PSZ. Again,
fin.² ZS : fin. ZPS :: fin. ZP : fin. PSZ, therefore,
fin.² SZ : fin. ZPS :: fin. ZP \times fin. ZS : fin. PSZ;
therefore by equality fin.² ZO : fin. ZPO :: fin.² ZS : fin. ZPS.
From Z, on the great circles PO, PS, demit the perpendiculars ZE, ZI, and lay off the arch FP = EP; make FH perpendicular

to OP and = to EZ, and join SH with a great circle. Then the triangles OZE, SFH are evidently equal in every respect, therefore the problem is reduced to this, viz.—From the given points Z, H, so to draw the arches ZS, HS to intersect in the circle FIS, that their sines may obtain a given ratio.

To effect this, put a and b for the sine and cosine of IF, m and n for the cosines of FH and ZI, p and q for the sines of ZPO and ZPS respectively, and x for the sine SI, the cosine of which will be $\sqrt{(1 - x^2)}$ to radius unity.

And by trigonometry the cosine of ZS = $n \sqrt{(1 - x^2)}$, the cos. SF = $b \sqrt{(1 - x^2)} - ax$, and cos. SH = $bm \sqrt{(1 - x^2)} - max$; therefore, sin.² SH = $1 - (bm \sqrt{(1 - x^2)} - max)^2$;

theref. by quest. $1 - n^2 (1 - x^2) : 1 - (bm \sqrt{(1 - x^2)} - max)^2 :: q : p$. Hence by reduction, putting $v = (qma)^2 - q(bm)^2 - pn^2$, $w = 2qbm^2a$, and $r = q(bm)^2 - pn^2 + p - q$, we get $vx^2 + wx \sqrt{(1 - x^2)} = r$, whence, by completing the square, x^2 is found = $(\frac{1}{2}w^2 + rv) \div (v^2 + w^2) \pm \sqrt{[(\frac{1}{2}w^2 + rv)^2 \div (v^2 + w^2)^2] - r^2 \div (v^2 + w^2)}$. From this SI becomes known, and SP is readily determined.

XIII. QUESTION 101, answered.

ANALYSIS, by Messrs. I. T. M'Doneld and Swale.

Fig. 175, pl. 12, let ACB be the Δ required to be constructed; the base AB and perp. CD of which are given; ABCE its circumscribing circle whose diameter is Cm; also let CKE be the line bisecting the vertical angle.

Then by sim. Δ s AC : Cm :: CD : CB,
and EC : Cm :: CD : CK;

theref. rect. ECK = rect. ACB = 2 rect. AKB, per question.

But rect. AKB = rect. EKC; theref. rect. ECK = 2 rect. EKC;

hence EC = 2EK, or EL = LN;

theref. EL is given, and a circle described through the three points A, E, B, will cut NC, drawn parallel to AB, in C the vertex of the Δ required.

SYNTHESIS, by Messrs. Harris, Lowry, and Nicholson.

Conf. Take AB = the given base and bisect it at right angles with the indefinite perp. FLN, then take LE, LN, each = to the given perpendicular, and through the three points A, E, B, describe a circle meeting NC, drawn \parallel to AB, in C; join AC, CB, and ACB is the Δ required.

Demon. Draw CD \perp to AB and join CE. Then AB is the given base, and CD = LN = the given perp.

I

And,

Hence, if AF be taken such that twice its square may be = the given sum of the squares — half the square of the diff. of the segments of the base, and AS be made = $\frac{1}{2}$ AF, and SD be drawn \perp to AF, meeting a semi-circle described thereon in D, AD will be = half the base of the Δ , from hence the Δ is readily determined.

For, $AS \cdot SF = SD^2 = 2AS^2$;
 therefore $AD^2 = 3AS^2$ and $DF^2 = 6AS^2$;
 hence $2AD = DF$, or $AF^2 = 3AD^2$;
 consequently $2AF^2 = s^2 - \frac{1}{2}d^2 = 6AD^2$.

Q. E. D.

In the same manner was this question answered by Mr. Gregory.

XVII. QUESTION 105, answered by Mr. Nicholson.

Conf. Find DE (fig. 245, pl. 16.) by Eu. 29, VI, and Simpson's Geometry, Prob. III. B. VI. such, that the rectangle contained under $2DE$ and the sum of twice the given diff. of the sides, twice the given segment and $2DE$, may be = to twice the diff. of the square of the base, and the square of the diff. of the sides. In ED produced take EA = to the given diff. and DC = to the given segment; make AB = to the given base and BC = to CE; so shall ABC be the Δ required.

Demon. Join BD; since AB = the given base, DC = the given segment, and AE = the given diff. of the sides; we have only to prove that BD is perp. to AC, in order to which, we have,

by *conf.* $2AB^2 - 2AE^2 = (2AE + 2CD + 2DE) \times 2DE$;
 theref. $AB^2 - AE^2 = 2AE \cdot ED + 2ED^2 + 2CD \cdot DE$,
 and $AB^2 + DC^2 = (AE + ED)^2 + (ED + DC)^2$,
 consequently $AB^2 - BC^2 = AD^2 - DC^2$;
 therefore BD is perpendicular to AC.

Q. E. D.

The same by Mr. Lowry.

Conf. Take CD = the given distance intercepted between the perp. and vertical \angle , DQ = the given diff. of the sides; draw DB perp. to CD, and take DL = CD; also draw CLP and apply thereto QP such, that QP^2 may be = to the sum of the squares of the base and segment DC; demit the perp. PE, and make EA = DQ. To DB apply AB = the given base, and join CB; so shall ACB be the Δ required.

Demon. By *conf.* AB = the base, and BC = the given seg. And, since CD = DL, CE is = EP, and AD = EQ;
 theref. $EC^2 + EQ^2 (AD^2) = QP^2 = AB^2 + DC^2$, by *conf.*

But

But $BC^2 + AD^2 = AB^2 + DC^2$;
 theref. $EC^2 + AD^2 = BC^2 + AD^2$, and $EC = BC$;
 hence $AC - BC = AE = DQ =$ the given diff. of the sides.
Q. E. D.

The same by Mr. Swale.

Analysis. Let us suppose the thing done, and ACB the Δ to be determined. Demit upon the opposite side AC the perp. BD; make Dd = DC, CE = CB, and join Bd, BE. Then, by Eu. II. 12,

$$\begin{aligned} AB^2 &= AE^2 + EB^2 + 2AE \cdot ED \\ &= AE^2 + Ed^2 + 2AE \cdot ED + 2Ed \cdot dD + EC^2 \\ &= AE^2 + (Ed + dD) \cdot Ed + EC^2 + 2AE \cdot ED \\ &= AE^2 + (Ed + EC) \cdot EC + 2AE \cdot ED \\ &= AE^2 + (EC + EA) \cdot 2ED. \end{aligned}$$

Whence $(EC + EA) \cdot 2ED = AB^2 - AE^2$.

Now make AF = dC, and EG = $\frac{1}{2}$ dC,

then $EC + EA = dF$, and $2ED = 2dG$.

Hence, $2dG \cdot dF = AB^2 - AE^2 =$ a given space, and FG is a given line.

Consequently the Prob. is reduced to the 29th Prop. of the 6th Book of Professor Playfair's Edition of Euclid's Elements.

Mr. M'Doneld also favoured us with an answer to this question.

XVIII. QUESTION 106, answered by Mr. Lowry.

Geometrical Analysis. Suppose it effected, and that ACB (fig. 246, pl. 16.) is really the Δ which was to be constructed. Let AHBG be the circumscribing circle, having its diameter perp. to the base AB. Join AH, CG, and demit the \perp s GD, HF, CK, CP and QB, the latter being perp. to CG; also join DK, IF, and IQ.

Then $RQ = QB$, and $AI = IB$,

theref. IQ is \parallel to AC, and $= AD = FC = \frac{1}{2}$ the given diff. of the sides; consequently IF is equal and parallel to QC.

In the same way it may be shewn that DK is \parallel to AH;

theref. $\angle DKI = \angle DFI = \frac{1}{2}$ the given vertical $\angle ACB$;

wherefore the points D, I, F, K are in a circle,

and consequently the rect. DEF = to the rect. IEK.

Again, because the ratio of AP to PB is given, the ratio of AI to IP, that is, the ratio of AE to EC, that is, the ratio of \angle E to EK, is given, and $IK = PC$ is given in magnitude; therefore the rect. DEF = the rect. IEK, is given.

Hence, there is given $AD = FC$, the ratio of AE to EC, and

the rectangle DEF to determine the lines AE, EC, which is done at question 61, Repository.

The same by Mr. Nitholfsen.

Conf. In fig. 247, pl. 16, take M to the given perp. in the given ratio of the segments of the base, the antecedent being greater than the consequent, and make the square upon S = to the diff. of the squares upon M and the perp. Upon AE = the given diff. of the sides erect the perp. AH = S, and join HE, and in AE produced take EI = the given perp. to HE, produced if necessary, apply IK = M, and draw HC || to KI to meet AE in C. Now upon AC describe a semi-circle, in which apply CD = the given perp. join AD, and produce it indefinitely; then to AD apply CB = CE, and ACB will be the Δ required.

Demon. By *Conf.* AE = AC — BC the given diff. and CD is perp. to AB and equal to the given one. Therefore we have only to prove that AD and DB have the given ratio.

By the construction M = IK, and CD = EI;
theref. by sim. Δ s, &c. CH : M :: CE : CD

or $AC^2 + M^2 = CD^2 : M^2 :: CB^2 : AC^2 - AD^2$;

and by division $AC^2 - CD^2 : M^2 :: DB^2 : CD^2$;

that is, AD : M :: DB : CD,

or, AD : DB :: M : CD.

Q. E. D.

Answers to this question were also received from Messrs. Mc. Doneld, Swale, and Thornoby.

XIX. QUESTION 107, answered by Mr. Harris.

Conf. From any point C (fig. 248, pl. 16.) draw Ca, Cb making the angle aCb with each other = the given vertical \angle ; take Ca of any length, and upon it constitute the parallelogram Cadb = to double the given area, then bisect the \angle aCb by the line COF, and take CO, CF such, that the rect. OCF may be = to the rect. aCb, and OF = the given distance between the centres. Bisect OF in E, and about the centre E, with the distance EO or EF describe a circle cutting the lines Ca, Cb, in A and B, and join AB; then ACB is the Δ required.

Demon. Since EA, EO, EB are equal, it is evident that O is the centre of the circle inscribed in the Δ ACB, and F the centre of the circle touching the base and the continuation of the sides CA, CB, and OF = the given distance.

Again, by *cor.* to Qu. 63, Rep. the rect. ACB = rect. OCF; but by *conf.* rect. OCF = rect. aCb; \therefore rect. ACB = rect. aCb, and theref. the area of the Δ ACB = area Δ aCb = $\frac{1}{2}$ the parall. Cadb = the given area by *conf.*

Q. E. D.

The

The same by Mr. Lowry.

Conf. Take OF (fig. 249, pl. 16.) = the given distance, and make $\angle FOD = \frac{1}{2}$ the vertical \angle . On OD drop the perp. FD, and continue DO, till the rect. ORD be to the given area as OD to FD, and complete the parallelogram RCDE, and from the centres O and F, describe two circles to touch CE as at P and E; draw the tangents AB, AC, and ABC is the Δ required.

Demon. OF is the given dist. and the $\angle ACB = \angle ACO + \angle BCO = 2FOD =$ the given angle.

Again, by sim. Δ s, $OD : FD :: OR : OP :: ORD : OP \cdot RD = OP \cdot CE =$ the Δ ABC:

but, by *conf.* $OD : FD ::$ rect. ORD : the given area, therefore the Δ ABC = the given area. Q. E. D.

This problem was constructed in a similar manner by Mr. Gregory.

The same answered by Mr. Nicholson.

Conf. Upon PL = the given distance as a hypotenuse, construct the right \angle Δ LGP (fig. 250, pl. 16.) having the $\angle P =$ half the given vertical one. Take $GM^2 =$ the given area, and GT to TP as GL to GP; join MT and produce it to meet PI drawn \perp to GM. Bisect MI in H, and GP produced apply HC = to HI, and draw CD \parallel to PL to meet IP, GO produced in O and D. Then with centres O and D and radii OP, GD describe the circles O and D, to touch which draw AB to meet CG in A, and CB (the tangent to both circles) in B; then ABC is the Δ required.

Demon. Upon the diameter MI describe a circle, which by the construction will pass through C; produce MG, CG to meet the circumference in K and E;

then by the circle, $EG = PC$, and $GK = PI$;

and therefore $GC \times CP = MG \times PI$;

but by sim. Δ s and *conf.* $GM : PI :: GT : TP :: GL : GP :: PO : PC$,

and by division, &c. $GM^2 = GC \times PO =$ the given area,

as it is well known that $2GC = AC + CB + AB$;

and by parallels $DO = LP =$ the given dist. Q. E. D.

The same by Mr. Swale.

Analysis. Let us suppose the Δ ACB (fig. 192, pl. 13.) to be that required, and the several lines drawn as in the figure. Then since $OS (= 2OI = 2AI = 2BI)$ and the $\angle ACB (= 2\angle IAB = 2\angle IBA)$ are given, $AI (= IB)$ and the $\angle IBA (= IAB)$ will be given; and consequently

frequently the $\triangle AIB$ will be readily determined. Now AB being known, and the rect. $AB \times CR =$ a given one, CR will be known, and consequently the $\triangle ACB$.

The Geometrical Analysis by Mr. M'Doneld, was very little different from the above.

XX. QUESTION 108, answered by Mr. Lowry.

Let (fig. 251, pl. 17.) be the place of the ring when the rod is in a vertical position, then $2AS$ will be equal to the difference between the length of the string and rod. Now let the rod be supposed to revolve with an uniform angular velocity about the point A till it comes to the position AF , and let C be the place of the ring at that time. Draw AC , FC , the latter meeting the vertical line BAR in R ; drop the perpendiculars FG , CQ , and continue the latter to meet FM , drawn parallel to AB in M . Then, by Mechanics, the angles QAC , CFH are equal, therefore, their supplements, the angles ACQ , TCH , or QCR are equal; consequently the right angled \triangle s RQC , AQC , will be equal in all respects, for the side QC is common to them both. This being premised, put $a = RF = AC + CF =$ the given length of the string, $b = AF = AB =$ the given length of the rod, $c = AS$, $x = SQ$ the abscissa, and $y = QC$ the corresponding ordinate of the required curve, then will $AQ = QR = x + c$, $RG + AG = 2(x + c + AG)$.

And by Trigonom. $RG + AG : RF + AF :: RF - AF : AR$, or, $2(x + c + AG) : a + b :: a - b : 2(x + c)$; therefore $AG = (a^2 - b^2) \div 4(x + c) - x + c = m^2 - 4(x + c)$, m^2 being put $= a^2 - b^2$, hence, $RG = AR + AG = 2AQ + AG = 2(x + c) + (m^2 - 4(x + c)) \div 4(x + c) = (4(x + c)^2 + m^2) \div 4(x + c)$.

Then by similar triangles $RG : RF :: x + c : AC$, or, $(4(x + c)^2 + m^2) \div 4(x + c) : a :: x + c : 4a(x + c)^2 \div (4(x + c)^2 + m^2) = AC$; but by Eu. I. 47. $AC^2 - AQ^2 = QC^2$; hence

$(4a(x + c)^2 \div (4(x + c)^2 + m^2))^2 - (x + c)^2 = y^2$, or by reduction,

$\left[\frac{4a}{m^2 \div x + c + 4} \right]^2 - x + c = y^2$, the equation of the required curve.

Remarks. If $x = 0$, y is $= 0$; and when $x = b$, or $x + c = BS = SV$, y is also $= 0$, so that the curve passes through S and V .

When $x = \frac{1}{2}m$ the rod will be parallel to the horizon, and y a maximum and $=$ to $\frac{1}{2}b$.

XXI. Or PRIZE QUESTION, 109, answered by Mr. Thornoby.

A triangle answering the conditions of the problem may be found thus. Let ASB (fig. 255, pl. 16.) be the circle given in magnitude and position, whose centre is O , and P the given point. Join PO and perp. thereto draw PS to meet the circle S ; join OS , and through P draw $APEB$ perp. thereto meeting the circle in A and B . Divide AB in Q so that $AP : PB :: AQ^2 : QB^2$, and on ES , (produced if necessary,) take $ED =$ the given perp. and draw CD parallel to AB ; then by the lemma on page 336, Simpson's Algebra, describe a circle to intersect CD in C so that if AC, BC , be drawn, AC may be to BC as AQ to QB , and ABC will be the Δ required.

For the base AB passes through the given point P and its extremities fall in the circumference of the given circle, and $ED = CI =$ the given perp. and by construction $AC^2 : BC^2 : AQ^2 : QB^2 :: AP : PB$.

The same by Mr. John Surtees, Sunderland.

The base AB (fig. 256, pl. 16.) being drawn any how through P the given point, then AP, PB and AB are given. Put $b = AE = EB$, $p = EH = CD$ the given perp. $n = AP$, $m = PB$ and $x = ED$. Then $(b + x)^2 + p^2 = AC^2$, $(b - x)^2 + p^2 = BC^2$, and per quest. $BC^2 \times m = AC^2 \times n$, or $\frac{m + n}{m - n} \times 2bx =$

$x^2 = p^2 + b^2$, or $\frac{AB^2}{2PE} \times x - x^2 = AH^2$. Hence this

Construction. Take $EL = 2EP =$ the difference between the segments of the base made by the given point, and EM a third proportional to LE and AB . Upon LM, EM let the semi-circles LGM, EFM be described, and erect the perp. EG and take therein $EH =$ the given perp. and join AH ; also in EG take $EI = AH$ and through I draw IF parallel to AB . Drop the perp. FD which will meet HC , drawn parallel to AB , in C the vertex of the Δ required; therefore join AC, CB , and ACB will be a Δ answering the conditions of the question.

Ingenious solutions were also received from Messrs. Harris, Hill, and Swale.

The Medal for solving the Mathematical Prize Question is decided in favour of Mr. S. Thornoby, who will please to send for it to Mr. Glendinning's, by whom it will be delivered, free from expence, to any part in London.

ARTICLE

ARTICLE XVIII.

MATHEMATICAL QUESTIONS,

To be answered in Number IX.

I. QUESTION 131, *by Mr. Thomas Milner, Teacher of the Mathematics at Lartington, near Barnard-Castle.*

IF my age in days be equal to the least whole number that can be divided by 15, 21, 27, and 29, the remainders will be 8, 11, 5, and 18 respectively. Query my age?

II. QUESTION 132, *by Mr. J. Collins, School-master, Kensington.*

Given the diameter of a semi-circle $= 24$, 'tis required to inscribe therein a parallelogram whose longest side shall be parallel to the diameter, so that the area of the same may be equal to the area of the greatest circle that can be inscribed in the remaining segment. Quere the dimensions of each?

III. QUESTION 133, *by Mr. Ralph Simpson, Croxdale.*

To find three numbers such, that not only the product of every two, but also the product of all three added to a given square number, shall make a square number.

IV. QUESTION 134, *by Mathesis, near Bath.*

It is required to draw a chord through a given point within a given circle, such, that the parts thereof intercepted by that point and the periphery may obtain a given ratio?

V. QUESTION 135, *by Mr. James Lee, London.*

Given the two sides, and the sum of the base and perp. of any plane Δ to describe it?

VI. QUESTION 136, *by Mr. Olinthus Gregory.*

The perimeter of an isosceles triangle is equal to $16 + 8\sqrt{5}$
and

and its vertical angle is 36° ; it is required to find the length of each side of this triangle without having recourse to trigonometrical tables?

VII. QUESTION 137, *by Mr. William Passman, Teacher of the Mathematics, at Hull.*

Of all cones having the same external surface, exclusive of the base, required that, which will vibrate quickest when suspended by the vertex?

VIII. QUESTION 138, *by Mr. B. Haynes, Land-Surveyor, at Salisbury.*

Being employed to plan an Estate, part of which consisted of a mountain of considerable height, bordered on one side by a plain, now I would request to be informed how I must proceed to lay down the said mountain properly in my plan, and also how I may find the angle of its elevation with the horizon, making use of a Gunter's Chain only in the field?

IX. QUESTION 139, *by Mr. John Harris.*

An horizontal dial being made for one place; to set it up in another place that has a difference of longitude from the first, so, that it may shew the true hour of the day, in the place it was made for.

X. QUESTION 140, *by Mr. W. Francis, Jun.*

There is a light-house built upon a rock, one side of which stands perpendicular, facing the sea shore, on the top whereof a pendulum, that measured 4.971 inches from the point of suspension to the centre of its circular bob of .6 inch radius, was observed to vibrate 50399 times in five hours; now admitting the earth to be a perfect globe 7958 miles in diameter, I would know the height of the light-house, being $\frac{3}{4}$ that of the rock; and also the distance of a privateer from the foot of the rock, at which I saw the fire from a gun, 20 vibrations of the pendulum, before the sound reached my ear?

XI. QUESTION 141, *by the Rev. Mr. L. Evans.*

If the given trapezoid ABCD (fig. 253, pl. 16.) right angled at B and C, be divided into two equal parts, by the line CF, drawn from the point E bisecting BC, it will be as $AB + CD : AD ::$
AB

AB : DF. But, if the line EF, be drawn from the point F bisecting AD, I say it will be as $AB + CD : BC :: AB : CE$.
Query Demonstration?

XII. QUESTION 142, by Mr. John Howard.

Given the segments of the base, and the sum of the sides to determine the great circle spherical triangle.

XIII. QUESTION 143, by Mr. Wm. Marrat.

If a tin can, in the form of a conical frustum, whose top diameter is $6\frac{1}{2}$, bottom diameter 8.7, and perp. altitude $11\frac{1}{2}$ inches respectively, be immersed in water with its greater end downwards, and its axis perp. to the surface, and it sink to the depth of 2 inches measured on the slant side : required the thickness of the tin ; the specific gravity of tin and water being 7300, and 1000 ?

XIV. QUESTION 144, by Mr. Samuel Thornoby.

It is required to cut a common playing card (the length being less than twice its breadth) into the three pieces so that when put together they may make a square.

XV. QUESTION 145, by Mr. Gregory.

The circumference of the circle AFKG (fig. 254, pl. 16.) is divided into an even number of equal parts, as AB, BD, DF, . . . GE, &c. and the dividing points are connected by parallel chords, as CB, ED, GF, &c. Now the sum of all the chords, is known to be $168 + 84\frac{1}{3}$, and the length of the greatest chord is 84 : from which it is required to determine the number of chords, and the length of each.

XVI. QUESTION 146, by Philo, Newcastle.

Most authors who have written on the theory of Water Mills make the ratio of the stream (in undershot wheels) to that of the water-wheel as 3 to 1 when the effect produced is the greatest possible : But Mr. Waring in the 3rd vol. of the American Philosophical Transactions, page 144, makes the ratio as 2 to 1. It is here required for the information of those concerned in the construction of such works, to investigate the truth or falsity of Mr. Waring's Theory.

XVII. QUESTION 147, by Mr. Haynes, Salisbury.

In a gentleman's park there is a piece of water on which I observed a small skiff at anchor, having two masts, the distance between which was known. By standing on the brink of the water in a line with the masts, I observed that a ray drawn from the top of the highest to the edge of the water cut the top of the lowest. Now being only possessed of an instrument for measuring vertical angles, in what manner must I proceed to obtain the breadth of the water in the direction of the masts, admitting that it is accessible on all sides?

XVIII. QUESTION 148, by Mr. I. H. Swale.

It is required to find a point P, such, that lines being drawn to it from the three angular points A, C, B, of a given triangle ACB, the sum of the solids $PA^3 \times AC$, $PB^3 \times BC$, and $PC^3 \times AB$ may be equal to a given solid.

XIX. QUESTION 149, by Mr. I. T. M'Doneld.

Given the base, the vertical angle, and the ratio which the line bisecting the vertical angle has to the difference of the segments of the base made thereby to construct the triangle.

XX. QUESTION 150, by Mr. William Peacock, Land-Surveyor, at Birmingham.

Given the base, the vertical angle, and the rectangle under the sides and the line bisecting the base to construct the plane triangle.

XXI. QUESTION 151, by Mr. John Lowry, Excise-Officer, Birmingham.

Given the line bisecting the base, and the line bisecting the vertical angle, to determine the triangle, when the solid, whose base is the square of the perpendicular and altitude the difference of the segments of the base made by the bisecting lines, is a maximum.

XXII. QUESTION 152, by Mr. Geo. Brown, Teacher of the Mathematics, at Howdon-pans, near North Shields.

Let AB, AC be two straight lines given by position, let B and C be given points in these lines, and let P be a given point without them.

them. It is required to draw two straight lines PD, PE meeting the given lines in D and E so that the angle DPE may be of a given magnitude, and so that BD may have to CE a given ratio.

XXIII. QUESTION 153, by Mr. James Wolfenden, *Hollinwood, near Manchester.*

In a given circle to inscribe a triangle such, that the triangle formed by joining the three points, where perpendiculars from the extremities of the base and vertical angle, meet the line bisecting that angle and the base respectively, shall be a maximum.

XXIV. QUESTION 154, by Mr. Wm. Wallace, *Perth.*

It is required to describe a triangle which may have its angles upon the circumferences of three given circles, and which may be similar to the triangle formed by straight lines joining their centres, and also, which may have one of its sides passing through a given point.

XXV. QUESTION 155, by Mr. James North.

To assign the correct fluents of the equation $\dot{x}e^{\frac{2x}{a}} = -\frac{\dot{z}}{v^3}$;

where v is the cosine of the arc or angle z , to the radius 1, and $e = 2.718281828$ the number whose hyp. log. is 1; x being $= 0$ when $z =$ the given arc b . The fluent to be in finite terms of the sine, cosine, and tangent of the arc z , and given quantities.

XXVI. PRIZE-QUESTION 156, by Mr. John Howard, *Mathematician, Newcastle-upon-Tyne.*

Suppose a sphere to descend by its own gravity along any given curve, placed in a plane that is perpendicular to the horizon; it is required to determine that point where it will quit the curve?

ARTICLE XIX.

MATHEMATICAL LUCUBRATIONS.

By Mr. WILLIAM WALLACE, *Assistant-Teacher of the Mathematics in Perth Academy.*

PROPOSITION I. THEOREM. *Fig. 258, Pl. 17.*

LET ABC be a triangle inscribed in a circle; from any point V in the circumference, let there be drawn perpendiculars VD, VE, VF to the sides of the triangle: the points D, F, E lie in a straight line.

Join AV, BV, CV; join also DF and EF. Because the angles VFC, VEC are right angles, a circle may be described about the quadrilateral VFCE: in like manner it appears that a circle may be described about the quadrilateral VFDA; therefore the angles VAD and VFD are together equal to two right angles; but, because VABC is a quadrilateral in a circle, VAD is equal to VCE, which, (since the points V, F, C, E are in a circle) is equal to VFE; therefore the angles VFE, VFD are together equal to two right angles; therefore DF and FE lie in a straight line.

Cor. The triangle DVE is similar to the triangle AVC, also DVF is similar to BVC, and EVF is similar to BVA.

PROP. II. LOCUS. *Fig. 259, Pl. 17.*

Let ABC be a given triangle, let a point V be taken so, that if VD, VE be drawn perpendicular to AB, BC the sides of the triangle, and VF perpendicular to AC the base, and FD, FE be joined, the ratio of FD to FE may be given; the point V shall be in the circumference of a given circle.

Join AV, CV and draw DG and BH perpendicular to AC. Because the angles ADV, AFV are right angles, the points A, D, V, F are in the circumference of a circle, therefore the angle VAD is equal to VFD, that is to the alternate angle FDG, now ADV, FGD are right angles, therefore the triangles ADV, DGF are similar, hence

$FD:AV=BG:DA=BH:BA$, and by alternation $FD:BH=AV:BA$: in like manner it appears that $BH:FE=BC:VC$. Therefore

$FD:FE=BC:AV:BA:CV$, and because the ratio of FD to FE is given by hypothesis, the ratio of $BC \cdot AV$ to $BA \cdot CV$ is given, and since BC, BA are given lines, the ratio of AV to CV is also given, now the points A and C are given, therefore, by a well known proposition, the point V is in the circumference of a given circle.

From the preceding analysis it appears that the given ratio of DF to FE must not be the same with the ratio of BC to BA, for in that case AV would be equal to VC and the point V would be in a given straight line.

The *locus* of V may be found by the following construction: Let the given ratio be that of P, a given line to AB a side of the triangle, but P must not be equal to BC: divide AC at K and k so that $P:BC=AK:KC=Ak:kC$; a circle described on Kk as a diameter shall be the locus required: that is, if from any point V in the circumference there be drawn perpendiculars VD, VE to the sides of the triangle and VF perpendicular to the diameter of the circle, and if FD, FE be joined, FD shall be to FE in the given ratio of P to AB. For the points K, k being found as in the preceding construction, it is well known that if straight lines be drawn from any point V in the circumference to the points A and C, $AV:VC=AK:KC=Ak:kC$, i. e. by const. $=P:BC$, and theref. $BC \cdot AV:BA \cdot CV=BC \cdot P:BA \cdot BC=P:BA$; and

and it has been shewn in the preceding analysis that $BC \cdot AV : BA \cdot CV = DF : EF$; therefore $DF : EF = P : AB$.

Q. E. D.

From this proposition the truth of the following *Local Theorem* is obvious.

PROP. III. LOCAL THEO. Fig. 259, 260, Pl. 17.

Let AB, CB be two straight lines given by position, meeting Kk the diameter of a given circle in A and C , so that $AK : KC = Ak : kC$; from any point V in the circumference let there be drawn VD, VE perpendicular to the given lines and VF perpendicular to the diameter of the circle and let DF and EF be joined. DF shall be to EF in the given ratio of $AK \cdot BC$ to $CK \cdot BA$.

Cor. If the given straight lines AB, CB meet at a point B in the circumference (fig. 260.) then $AK : KC = AB : BC$ and $AK \cdot BC = AB \cdot CK$, therefore also $FD = FE$.

In the same way that Prop. II. is derived from a well known circular locus, may the following propositions concerning LOCUS be derived from other known propositions; their demonstrations therefore are not added, but left as exercises to the Student in Geometry.

PROP. IV. LOCUS. Fig. 261, Pl. 17.

Let ABC be a given triangle; let a point V be taken, such, that, if VF be drawn perpendicular to BC , the base of the triangle; and VD, VE perpendicular to AB, AC the sides; and FD, FE be joined, the difference between the squares of DF and EF may be equal to a given space; the point V shall be in the circumference of a given circle.

K 3

PROP.

PROP. V. LOCUS, Fig. 261, Pl. 17.

Let ABC be a given triangle ; let V be so taken, that perpendiculars being drawn as in the last proposition, and also FD , FE being joined, the sum of the squares of FD , and FE may be equal to a given space ; the point V shall be in the circumference of a given circle.

PROP. VI. LOCUS. Fig. 262, Pl. 17.

Let ABC be a given triangle, let a point V be so taken, that, if VD , VE , VF be drawn perpendicular to the sides of the triangle, and the points D , E , F be joined, the sum of the squares of the sides of the triangle DEF may be equal to a given space ; the point V is in the circumference of a given circle.

A similar proposition may be extended to the figure formed by drawing perpendiculars to the sides of any rectilineal figure whatever.

From the conversion of these *LOCUS* elegant *LOCAL THEOREMS* as well as a number of curious *PORISMS* may evidently be formed : but these, as well as the solutions of the two following problems, we shall omit at present, recommending them however to the consideration of the Student in Geometry.

Prob. 1. To find a point such that if perpendiculars be drawn from it to the sides of a given triangle, the straight lines which join the points where these perpendiculars meet the sides shall form a triangle similar to a given triangle.

Prob. 2. Fig. 259, Pl. 17. To find a point V in the circumference of a given circle, such, that if VD , VE be drawn perpendicular to two straight lines given by position, and VF perpendicular to the diameter of the circle, and FD , FE be joined, the

the rectangle $FD \cdot FE$ may be equal to a given space; also to determine the point V so that the rectangle may be the greatest possible.

PROP. VII. PROBLEM.

Having given the sum of the sides of a triangle, as also the sum of their squares, and either the sum of their cubes, or the solid contained by the three sides, to find the area of the triangle.

Let the sides of the triangle be a, b, c .

$$\begin{aligned} \text{Let } A &= a + b + c, \\ B &= a^2 + b^2 + c^2, \\ C &= a^3 + b^3 + c^3. \end{aligned}$$

The relation between the area of a triangle and its sides is expressed by this equation,

$$\frac{A}{2} \cdot \left(\frac{A}{2} - a\right) \cdot \left(\frac{A}{2} - b\right) \cdot \left(\frac{A}{2} - c\right) = (\text{area})^2, \text{ that is,}$$

$$\frac{A^4}{16} - (a+b+c)\frac{A^3}{8} + (ab+ac+bc)\frac{A^2}{4} - abc\frac{A}{2} = (\text{area})^2.$$

$$\text{Now } a + b + c = A,$$

$$\text{and } ab+ac+bc = \frac{(a+b+c)^2 - (a^2+b^2+c^2)}{2} = \frac{A^2 - B}{2},$$

$$\text{also } abc = \frac{a^3+b^3+c^3 + (a+b+c)(ab+ac+bc - a^2 - b^2 - c^2)}{3}$$

$$= \frac{2C - 3AB + A^3}{6}$$

$$\text{hence } (\text{area})^2 = \frac{A^4 - 2A^2B - 8Aabc}{16}$$

$$\text{or } (\text{area})^2 = \frac{-A^4 + 6A^2B - 8AC}{48}$$

and

$$\text{and the area} \left\{ \begin{aligned} &= \frac{\sqrt{A^4 - 2A^2B - 8Aabc}}{4} \\ &= \sqrt{\frac{-A^4 + 6A^2B - 8AC}{48}} \end{aligned} \right.$$

COROLLARY. Hence in any triangle,

$$\text{The radius of the inscribed circle} \left\{ \begin{aligned} &= \frac{\sqrt{A^3 - 2AB - 8abc}}{2A} \\ &= \sqrt{\frac{-A^3 + 6AB - 8C}{12A}} \end{aligned} \right.$$

PROP. VIII. THEO. Fig. 263, 264, Pl. 17.

Let ABCD be a quadrilateral described in a circle, let two of its opposite sides AC, BD, when produced, meet at E, let the diagonals AD, BC be drawn, intersecting each other at F, and let EF be joined, meeting AB at H. AH is to HB as the rectangle AC · AE to the rectangle BD · BE.

Draw LEK parallel to AB, meeting BC, AD produced in L and K.

Then AB : EL = AC : CE,

and EK : AB = ED : DB,

therefore EK : EL = AC · ED : CE · DB.

But ED : EC = AE : EB,

therefore AC · ED : EC · DB = AC · AE : EB · DB,

therefore EK : EL, or, AH : HB = AC · AE : EB · DB.

Q. E. D.

PROP. IX. PROBLEM. Fig. 265, Pl. 17.

Let ABC be a given triangle, a straight line FG, given by position, may be found, such, that if from any

any point V in AC one of the sides of the triangle, there be drawn VE , VD parallel to AB , BC , the other two sides, so as to form the parallelogram $VEBD$, and the diagonal DE be drawn, meeting FG , the line which may be found, in H ; the diagonal DE shall be divided at H into segments DH , HE , having to each other a given ratio.

Suppose the porism true, and that FG , the line to be found, meets AB , BC in K and L ; let DV meet KL in N . Because the ratio of DH to HE is given, the ratio of DN to LE is also given, and since this must be universally true, it is evident that if the point V were taken in AC , so that the point D might coincide with K , then also E would coincide with L : hence it appears that KL is the diagonal of a parallelogram $KPLB$ inscribed in the triangle ABC .

Now the given ratio of DH to HE , or of DN to LE , is compounded of the ratios of BL to LE and of DN to BL , that is of the ratios of AP to PV and of DK to BK , or of the ratios of AP to PV and of PV to PC ; therefore $DH : HE = AP : PC$, but the ratio of DH to HE is by hypothesis given, therefore the ratio of AP to PC is given, and, the line AC being given, the point P is also given.

Hence the following construction.

Let AC be the side of the triangle in which the point V is always to be taken, and let the given ratio of DH to HE be that of α to β ; Find P in AC so that $AP : PC = \alpha : \beta$; draw PK parallel to BC , meeting AB in K , and PL parallel to AB , meeting BC in L ; draw the diagonal KL , which will be the line that may be found, that is, if from V any point in AC there be drawn VD , VE , parallel to the other sides of the triangle, so as to form the parallelogram $DVEB$, and the diagonal DE be drawn, meeting KL in H ; $DH : HE = AP : PC = \alpha : \beta$.

The

The Synthetic Demonstration is evident from the preceding Analysis.

This Porism may be rendered more general as follows.

PROP. X. PORISM, Fig. 266, Pl. 17.

Let ABC be a given triangle, a straight line FG , given by position, may be found, such, that if from V , any point in AC a side of the triangle, there be drawn VD , VE , meeting the other sides AB , BC and making with them given angles at D and E , and if DE be joined, meeting FG , the line which may be found, in H , the straight line DE shall be divided at H into segments DH , HE having to each other a given ratio.

Let FG , the line to be found, meet AB and BC in K and L : By taking a particular case of the porism, as in the last proposition, it appears that when the point D coincides with K , then also E must coincide with L ; hence it follows that straight lines KP , LP , drawn parallel to DV , EV , must meet at P , a point in AC . Draw BQ parallel to DV or KP , meeting AC in Q , and draw QR parallel VE or PL meeting BC in R ; because the angle ABQ is given, the point Q is given; and because the angle QRC is given, the point R is also given. Draw DN parallel to BC meeting KL in N , then, $DH : HE = DN : LE$; now, the ratio of DN to LE is compounded of the ratios of BL to LE , and DN to BL ; and $DN : BL = DK : BK = VP : PQ = LE : ER$; therefore the ratio of DN to LE is compounded of the ratios BL to LE and of LE to ER , wherefore $DN : EL = BL : LR$. But the ratio DN to LE , or of DH to HE is by hypothesis given, therefore the ratio of BL to LR is given, and, the points B , R being given, the point L is also given; and because LP , PK make given angles with BC ,

BC, BA, the point K is given, and the straight line KL is given by position, as was to be shewn.

Construction. Let the given ratio be that of α to β ; draw BQ, meeting AC in Q, so that the angle ABQ may be equal to the angle which VD is to make with AB, and draw QR, meeting BC in R, so that the angle QRC may be equal to the angle which VE is to make with BC, take L in BR, so that $BL : LR = \alpha : \beta$, draw LP parallel to RQ, meeting AC in P, and draw PK parallel to BQ, meeting AB in K, through the points K, L draw the straight line FG, which is the line to be found; that is, if from V any point in AC, straight lines VD, VE be drawn parallel to PK, PL meeting AB, BC in D and E, and DE be joined, meeting KL in H, DH shall be to HE in the given ratio of α to β .

The Demonstration evidently follows from the Analysis, and for the sake of brevity is here omitted.

(To be continued.)

ARTICLE XX.

Extracts from a Paper on the Trigonometrical Tables of the Brahmins.

By JOHN PLAYFAIR, F. R. S. Edin.

And Professor of Mathematics in the University of Edinburgh.

(From the Transactions of the Royal Society of Edinburgh, Vol. IV.)

IN the second volume of the Asiatic researches an extract is given from the *Surya Siddhanta*, the ancient book which has been long, though obscurely, pointed out as the source of the astronomical knowledge of the Brahmins. The *Surya Siddhanta* is in the Sanscrit language: It is one of the *Sastras* or inspired writings of the Hindoos, and is called the *Jyotish* or Astronomical *Sastra*. It professes, as we learn from Mr. DAVIS, the ingenious translator,

translator, to be a revelation from heaven, communicated to MEYA, a man of great sanctity, about four millions of years ago, toward the close of the *Satya Yuga*, or of the golden age of the Indian mythologists, a period at which man is said to have been incomparably better than he is at present; when his stature exceeded twenty-one cubits, and his life extended to ten thousand years.

Interwoven, however, with all these extravagant fictions, this singular book contains a very sober and rational system of Astronomical calculations; and even the principles and rules of Trigonometry, a science of all others the most remote from fable, and the least susceptible of poetical decoration.

The circumference of the circle is here divided into 360 equal parts, each of which is again divided into 60, and so on; the same division was followed by the Greek mathematicians, and this coincidence is the more to be remarked, as it relates to a matter of mere arbitrary arrangement, and one by no means necessarily connected with the properties of the circle. Since this agreement cannot well be attributed to chance, it might be supposed to result from some communication having taken place between the two nations, were it not that another very probable cause may be assigned for it. In Greece, and no doubt in every other country, the division of the circle, into equal parts, is of a much older date than the origin of trigonometry, and must be as ancient as the first circular instruments used for measuring angles in the heavens. The inventors of these instruments naturally sought to make the divisions on them correspond to the space which the sun described daily in the ecliptic; and they could easily discover, without any very precise knowledge of the length of the year, that this might be nearly effected by making each of them the 360th part of the whole circumference. This principle may therefore have directed the astronomers, both of the East and West, to the same division of the circle, without any intercourse having taken place between them.

The next thing to be mentioned is also a matter of arbitrary arrangement, but one in which the Brahmins follow a method peculiar to themselves. They express the radius of the circle in parts of the circumference, and suppose it equal to 3438 minutes, or 60ths of a degree. In this they are quite singular. Ptolemy, and the Greek mathematicians, after dividing the circumference, as we have already described, supposed the radius to be divided into 60 equal parts, without seeking to ascertain in this division any thing of the relation of the diameter to the circumference; and thus throughout the whole of their tables, the chords are expressed in sexagesimals of the radius, and the arches in sexagesimals of the circumference. They had, therefore, two measures, and two units; one for the circumference, and another for the diameter.

The

The Hindoo mathematicians, again, have but one measure and one unit for both, *viz.* a minute of a degree, or one of those parts whereof the circumference contains 21600. From this identity of measures they derive no inconsiderable advantage in many calculations, though it must be confessed, that the measuring a straight line, the radius or diameter of a circle, by parts of a curve line, namely, the circumference, is a refinement not at all obvious, and has probably been suggested to them by some very particular view which they have taken, of the nature and properties of the circle. The measure here assigned to the radius, *viz.* 3438 of the parts of which the circumference contains 21600, is true to the nearest minute, and this is all the exactness aimed at in these trigonometrical tables; the author of them, however, must have known the ratio of the diameter to the circumference to a greater degree of exactness, for it appears from the Institutes of AKBAR * that the Brahmins knew it to be that of 1 to 3.1416, which is much nearer the truth than the preceeding. Calculating, as we may suppose, by this, or some other proportion, not less exact, the authors of the tables found, that the radius contained in truth 3437'. 44". 48". &c. and as the fraction of a minute is here more than a half, they took, as their constant custom is, the integer next above, and called the radius 3438 minutes.

These Trigonometrical tables are two, the one of sines and the other of versed sines. The sine of an arch they call *cramajya* of *jyapinda*, and the versed sine *utcramajya*, they also make use of the cosine or *bhujajya*. These terms seem all to be derived from the word *jya*, which signifies the chord of an arch, from which the name of the radius, or sine of 90°, *viz.* *trijya*, is also taken. This regularity in their trigonometrical language is a circumstance not unworthy of remark. But what is of more consequence to be observed, is, that the use of sines, as it was unknown to the Greeks; who calculated by the help of the chords, forms a striking difference between the Indian trigonometry and theirs. The use of the sine, instead of the chord, is an improvement which our modern trigonometry owes, as we have hitherto been taught to believe, to the Arabs; and it is certainly one of the acquisitions which the mathematical sciences made, when, on their expulsion from Europe, they took refuge in the East. But whether the Arabs are the authors of this invention, or whether they themselves received it, as they did the numerical characters, from India, is a question, which a more perfect knowledge of Hindoo literature will probably enable us to resolve.

No mention is made in this trigonometry of tangents, or secants; a circumstance not wonderful, when we consider that the use of

VOL. II. L these

* See Transactions R. S. Edin. Vol. II. page 135, Physical Class.

these was introduced in Europe no longer ago than the middle of the sixteenth century. It is on the other hand not a little singular, that we should find a table of versed sines in the *Surya Siddhanta*; for neither the Greek nor the Arabian mathematicians had any such, nor had we in modern Europe, till after the time of PITISCUS, who wrote about the end of the century just mentioned.

Next, as to the extent and accuracy of these tables. The first of them exhibits the sines to every twenty-fourth part of the quadrant, that is, the sine of $3^{\circ} 45'$, and of all the multiples of that arch, viz. $7^{\circ} 30'$, $11^{\circ} 15'$, &c. up to 90° . The table of versed sines does the same. In each, the sine, or versed sine, is expressed in minutes of the circumference, but without any fractions of a minute; and when the fraction that ought to have been set down is greater than $\frac{1}{2}$, the integer next greater is placed in the table. Thus the sine of $3^{\circ} 45'$ being, when accurately expressed in their way, $224' 49''$, is put down $225'$; and so of the rest. The numbers, therefore, are only so far exact as never to differ more than half a minute from the truth, and this very limited degree of accuracy gives, no doubt, to their trigonometry the appearance of an infant science. But when, on the other hand, we consider the principles and rules of their calculations, rather than the numbers actually calculated, we find the marks of a science in full vigour and maturity: and we will acknowledge, that the Hindoo mathematicians did not satisfy themselves with the degree of accuracy above mentioned, from any incapacity of attaining greater exactness.

Their rules for constructing their tables of sines may be reduced to two, viz. the one for finding the sine of the least arch in the table, that of $3^{\circ} 45'$, and the other for finding the sines of the multiples of that arch, its triple, quadruple, &c.

With respect to the first, the method proceeds by the continual bisection of the arch of 90° , and correspondent extractions of the square root, to find the sine and cosine of its half, its fourth part and so on. The rule, when the sine of an arch is given, to find that of half the arch, is precisely the same with our own: "The sine of an arch being given, find the cosine, and thence the versed sine of the same arch; then multiply half the radius into the versed sine and the square root of the product is the sine of half the given arch." Now as the sine of 90° was well known to these mathematicians to be half the radius, from thence was found the sine of 45° , then the sine of $22^{\circ} 30'$, and lastly of $11^{\circ} 15'$, which is the sine required. Thus the sine of $3^{\circ} 45'$ would be found equal to $224' 44''$, or 225 nearly, and the sine of $7^{\circ} 30'$ equal to $448' 89''$, or 449 nearly.

When

When the sine of $3^{\circ} 45'$, or of $225'$, was found equal to 225, the rest of the table was constructed by a rule, that for its simplicity and elegance, as well as for some other reasons, is entitled to particular attention. It is as follows " Divide the first jyapinda, 225' by 225; the quotient 1, deducted from the dividend, leaves 224', which added to the first jyapinda, or sine, gives the second, or the sine of $7^{\circ} 30'$, equal to 449'. Divide the second jyapinda, which is thus found, by 225, and deduct 2, the nearest integer to the quotient, from the former remainder 224', and the new remainder 222', added to the second jyapinda, will give the third jyapinda equal to 671'. Divide this last by 225, and subtract 3, the nearest integer to the quotient, from the former remainder 222', and there will be left 219, which added to the third jyapinda, gives the fourth; and so on to the twenty-fourth or last.

It is not immediately obvious on what geometrical principle this rule is founded, but a slight change in the enunciation will remove the difficulty. The rule may be expressed more generally thus: Divide any sine by 225, and subtract the quotient, or the integer nearest the quotient, from the difference between that sine and the sine next less; the remainder is the difference between the same sine and the sine next greater: and, therefore if it be added to the former, will give the latter. If then (fig. 267, pl. 17.) GA, GC, GE be three contiguous arches in the table, of which the differences AC, CE of consequence are equal, and of which the sines are AB, CD, and EF, the rule as last stated gives us, $CD - AB =$

$$\frac{CD}{225} = FE - CD, \text{ hence } EF + AB = CD \left(2 - \frac{1}{225} \right) = CD$$

$\left(\frac{449}{225} \right)$. But 225 is the sine of the arch $3^{\circ} 45'$, and 449 of twice,

that arch, as already shewn, and therefore, according to this rule if there be three arches, of which the common difference is $3^{\circ} 45'$, the sine of the mean arch will always have to the sum of the sines of the extreme arches a given ratio, that namely, which the sine of $3^{\circ} 45'$ has to the sine of twice $3^{\circ} 45'$, or of $7^{\circ} 30'$; now this is a true proposition; and, therefore, we are in possession of the principle on which the Hindoo canon is constructed. The geometrical theorem which is thus shewn to be the foundation of the trigonometry of Hindostan, may also be more generally enunciated. " If there be three arches in arithmetical progression, the sine of the middle arch is to the sum of the sines of the two extreme arches, as the sine of the difference of the arches to the sine

of twice that difference." This theorem is well known in Europe, and is justly reckoned a very remarkable property of the circle.

Now it is worth remarking that this property of lines, which has been so long known in the East, was not observed by the mathematicians of Europe till about two hundred years ago. The Theorem indeed concerning the circle, from which it is deduced under one shape or another, has been known to them from an early period, and may be traced up to the writings of EUCLID, where a proposition nearly related to it forms the 97th of the Data: "If a straight line be drawn within a circle given in magnitude, cutting off a segment containing a given angle, and if the angle in the segment be bisected by a straight line produced till it meet the circumference: the straight lines, which contain the given angle, shall both of them together have a given ratio to the straight line which bisects the angle." This theorem differs from the Hindoo rule, only by affirming a relation to hold of their lines. It is given by EUCLID as useful for the construction of geometrical problems; and trigonometry being then unknown, he probably did not think of any other application of it. But what may seem extraordinary is, that when about 400 years afterwards, PTOLEMY, the astronomer, constructed a set of trigonometrical tables, he never considered Euclid's theorem, though he was probably not ignorant of it, as having any connection with the matter in hand. He, therefore, founded his calculations on another proposition, containing a property of quadrilateral figures inscribed in a circle, which he seems to have investigated on purpose, and which is still distinguished by his name. This proposition contains, in fact, EUCLID's, and of course the Hindoo theorem, as a particular case; and though this case would have been the most useful to PTOLEMY, of all others, it appears to have escaped his observation; on which account he did not perceive that every number in his tables might be calculated from the two preceding numbers, by an operation extremely simple, and every where the same; and therefore his method of constructing them is infinitely more operose and complicated than it needed to have been.

Not only did this escape PTOLEMY, but it remained unnoticed by the mathematicians, both Europeans and Arabians, who came after him, though they applied the force of their minds to nothing more than to Trigonometry, and actually enriched that science by many valuable discoveries. They continued to construct their tables by the methods PTOLEMY had employed, till about the end of the sixteenth century, when the theorem in question, on which the Hindoo rule is founded, was discovered by VIETA; it is published in his *Treatise on Angular Sections*, not with his own demonstration, but

but was one given by an ingenious mathematician of our own country, ALEXANDER ANDERSON, of Aberdeen. It was then regarded as a theorem entirely new, and I know not that any of the geometers of that age remarked its affinity to the propositions of EUCLID and PROCLUS. It was soon after applied in Europe, as it had been for so many ages before in Hindostan, and quickly gave to the construction of the trigonometrical canon, all the simplicity it seems capable of attaining. From all this I think it might fairly be concluded, even if we had no knowledge of the antiquity of the Surya Siddhanta, that the trigonometry contained in it is not borrowed from Greece or Arabia, as its fundamental rule was unknown to the geometers of both those countries, and is greatly preferable to that which they employed.

ARTICLE XXI.

ATWOOD'S INVESTIGATIONS ON WATCH BALANCES.

(Continued from Page 363, Vol. I.)

THE vibration of a balance impelled by a single spiral spring only, has been the subject of the preceding investigations; but cases occur in which two or more springs are employed in giving vibratory motion to the balances of watches. Not to mention preceding instances, Mr. Mudge, an eminent watch-maker of the present times, has invented a method of combining the action of spiral springs, to impel the balance in each semiarc of vibration, on a principle not more remarkable for the novelty than it is for the ingenuity of the contrivance. The consideration of this additional case will therefore not be thought foreign to the present subject, especially as it may contribute to elucidate some circumstances respecting the effect of springs on the vibrations of balances, which at the first view are not at all obvious.

Let

Let two spiral springs be applied to act on a watch balance in the same direction ; if the two springs in unwinding themselves, by turning the balance come to the same point of quiescence, or, in other words, if the accelerative forces of both springs cease at the middle point of the vibration, whatever be the relative strength of the two springs, they will act on the balance precisely in the same manner as if one spring only had been applied, of equal strength with both (the springs being here supposed similar, in respect of the law of the elastic forces and tensions.) But when the points of quiescence of the two springs do not coincide, that is, when one spring continues to accelerate the balance in its vibration, after the acceleration of the other spring has ceased, the time of a semivibration must be obtained from a separate investigation.

Let the circumference of a balance (fig. 82, pl. 5.) be impelled by the action of a spiral spring through the semiarc of vibration BO, the forces of this spring being always in proportion to the angular distances from the point of quiescence O ; let a secondary or auxiliary spring also act on the balance from the extremity of the semiarc B as far as the point Q, at which point all acceleration of the auxiliary spring ceases, the forces of the auxiliary spring varying as the angular distances from the point of quiescence Q. Suppose that the accelerative force of the principal or balance spring on the circumference of the balance at the distance from quiescence OD is $= f$, take $Qd = OD$, and let the accelerative force of the balance spring on the circumference of the balance be to that of the auxiliary spring, when both springs are wound to the same angle $OCD = QCd$, in the proportion of 1 to n ; then the accelerative force of the auxiliary spring at the angular distance from

from quiescence Qd will $\equiv nf$; let $BO \equiv b$, $BQ \equiv c$, $QO \equiv d \equiv b - c$; also let OD or $Qd \equiv a$.

Suppose the balance to have described the arc BH by the joint action of both the springs, and let the arc BH be represented by x . Then because the accelerative force of the principal or balance spring at the angular distance from quiescence $OD \equiv a$ is f ; the accelerative force of the same spring at the distance OH is $\equiv f(b - x) \div a$; and since the force of the auxiliary spring at the angular distance from quiescence $Qd \equiv a$ is nf , the accelerative force of this spring at the angular distance from quiescence QH will be $\equiv nf(c - x) \div a$; wherefore the joint force of both springs to accelerate the circumference when at the distance OH from the point of quiescence O, that is, when the balance has described the arc BH, will be $\equiv (f \div a) \times (b - x + cn - nx)$. Let u be the space through which a body falls freely from rest by the acceleration of gravity, to acquire the velocity of the circumference when it has described the arc BH; this will give the following equation $\dot{u} \equiv (f \div a) \times (bx - x\dot{x} + cnx - nx\dot{x})$; and $u \equiv (f \div 2a) \times (2bx - x^2 + 2cnx - nx^2)$; and if $l \equiv 193$ inches, the velocity of the circumference when the arc BH has been described, is $\equiv \sqrt{(2lf \div a) \times (2bx - x^2 + 2cnx - nx^2)}$. Let t represent the time in which the balance describes the arc BH; then $t \equiv \sqrt{(a \div 2lf) \times \dot{x} \div (2bx - x^2 + 2cnx - nx^2)}$ and the fluent or $t \equiv \sqrt{(2a \div lf(n + 1)) \times}$ into a circular arc of which the sine is $\sqrt{(x(n + 1) \div (2b + 2nc))}$ to radius $\equiv 1$; which is the time of describing the arc BH. When $x \equiv c$, the time of describing the arc BQ, is $\sqrt{(2a \div lf(n + 1)) \times}$ into a circular arc of which the sine $\equiv \sqrt{(c(n + 1) \div (2b + 2nc))}$.

The time of describing the remaining part of the semicircle QO, (fig. 82, pl. 5.) is next to be determined,

mined. In the cases to which this investigation is applied, the auxiliary spring ceases entirely to act on the balance after it has described the arc BQ. This being stated, while the balance describes the remaining arc of the semivibration QO, it will be impelled by the balance spring only. To ascertain the time of describing the arc QO, it is first to be observed, that when the circumference has described the arc BQ, it will have acquired a velocity equal to that of a body which has fallen from rest by the acceleration of gravity through a space $\equiv (f \div 2a) \times (2bc - c^2 + nc^2)$.

Suppose the balance to have proceeded through the arc QR, (fig. 82, pl. 5.) and let $OR = x$; the force by which the circumference is accelerated at R $\equiv fx \div a$; and if u is the space through which a body falls freely from rest by the acceleration of gravity to acquire the velocity of the circumference in the point R, $\dot{u} = -fx \div a$; taking the fluents so that u may become $\equiv (f \div 2a) \times (2bc - c^2 + nc^2)$ when $x = d$, $u = (f \div 2a) \times (2bc - c^2 + nc^2 + d^2 - x^2)$; or because $d^2 = b^2 - 2bc + c^2$; $u = (f \div 2a) \times (b^2 + nc^2 - x^2)$; and the velocity of the circumference at R (fig. 82, pl. 5.) $\equiv \sqrt{(2lf \div a) \times (b^2 + nc^2 - x^2)}$; let t be the time of describing QR; then $t = \sqrt{(a \div 2lf)} \times x \div \sqrt{(b^2 + nc^2 - x^2)}$, and $t = \sqrt{(a \div 2lf)} \times$ into a circular arc of which the cosine is $x \div \sqrt{(b^2 + nc^2)}$ to radius $\equiv 1$, which should $\equiv 0$ when $x = d$; wherefore t or the time of describing QR $\equiv \sqrt{(a \div 2lf)}$
X

* When the circumference has described the arc BH $\equiv x$, it will have acquired a velocity equal to that of a body which has fallen freely from rest by the acceleration of gravity through a space $\equiv (f \div 2a) \times (2bx - x^2 + 2cnx - nx^2)$, as appears from the investigation in page 127; and when $x = c$, this space becomes $(f \div 2a) \times (2bc - c^2 + 2cn - nc^2) = (f \div 2a) \times (2bc - c^2 + nc^2)$.

\times into a circular arc of which the cosine is $\frac{x}{\sqrt{(b^2 + nc^2)}} = \frac{\sqrt{(a \div 2lf)}}{\sqrt{(b^2 + nc^2)}}$ into a circular arc of which the cosine is $\frac{d}{\sqrt{(b^2 + nc^2)}}$; and when $n = 0$, that is, when the entire arc QO has been described, the time $t = \frac{\sqrt{(a \div 2lf)}}{\sqrt{(b^2 + nc^2)}}$ into a circular arc of which the sine is $\frac{d}{\sqrt{(b^2 + nc^2)}}$.

The result of this investigation is, that the time in which the balance describes the femiarc BO, by the joint action of both springs through the arc BQ, and by the action of the balance spring only through

the arc QO, is $= \sqrt{\frac{2b}{lf(n+1)}} \times$ an arc, of

which the sine is $\sqrt{\frac{c(n+1)}{2b+2nc}} + \sqrt{\frac{n}{2lf}} \times$ an

arc, of which the sine is $\frac{d}{\sqrt{b^2 + nc^2}}$ (to radius

$= 1$) expressed in parts of a second*.

* When $n = 0$, this expression becomes $\sqrt{(a \div 2lf)} \times$ (twice the arc, of which the sine is $\sqrt{(c \div 2b)} +$ an arc, of which the sine is $d \div b$); but since $c = b - d$, the two arcs here mentioned will be exactly $= 90^\circ = p \div 2$; and the time of a semivibration $= \sqrt{(a \div 2lf)} \times p \div 2 = \sqrt{(ap^2 \div 8lf)}$, agreeing with the solution in page 162, Vol. I. Suppose, $d = 0.2$ since in this case $b = c$, the time of a semivibration becomes $\sqrt{(2a \div lf(n+1))} \times$ an arc, of which the sine is $\sqrt{(b \div 2b)}$ which arc is $= 45^\circ$, or $p \div 4$; wherefore the time of a semivibration in this case $= \sqrt{(ap^2 \div 8lf(n+1))}$, which is the true value, according to the solution in page 162, Vol. I. See, also page 136, of the present volume.

If the points of quiescence are in the first femiarc of vibration, and $c = 0$, the point B will coincide with Q (fig. 82, Pl. 3.), from which point the vibration will commence; in this case the expression for the time of a semivibration will become $t = \sqrt{(a \div 2lf)} \times$ an arc of which the sine is $d \div d$, or an arc of $90^\circ = p \div 2$, or $t = \sqrt{(ap^2 \div 8lf)}$; which agrees entirely with the solution in page 162, Vol. I. for in this case the auxiliary spring not acting on the balance while it describes the arc QON, the balance will vibrate

This solution is confined to that case in which the point of quiescence Q (fig. 82, pl. 5.) of the auxiliary spring is situated in the first semiarc of the vibration, that is, between B and O . Another case still remains to be considered, which is, when the point of quiescence Q of the auxiliary spring deviates from O by the given angular distance OQ , but is situated in the latter semiarc of vibration (fig. 83, pl. 5.), between O and E , instead of between O and B , as in the former solution. According to this condition, making $BQ = c$, $BO = b$, and the other notation remaining as before, it appears from an investigation no ways differing from the preceding, that the time in which the balance describes the semiarc BO will be $t = \sqrt{(2a \div f(n+1))} \times$ into a circular arc, of which the sine is $\sqrt{(b(n+1) \div (2b + 2ac))}$ expressed in parts of a second.

This result expresses the time in which the balance describes the semiarc BO , (fig. 83, pl. 5.) by the accelerative force of two springs, namely, the balance spring, of which the point of quiescence is O , and an auxiliary spring, of which the point of quiescence is Q . On considering this case more fully, when applied to the actual vibrations of a balance, it will appear evident, that the action of a third spring on the balance while it is describing the semiarc BO , must be taken into the calculation, in addition to the two springs already mentioned, in order to obtain a solution entirely correspondent with the circumstances of the case, when the points of quiescence of the auxiliary springs are situated in the latter semiarcs of the vibrations.

To state this more clearly, it is to be observed, that

vibrate by the force of the balance spring only; of which the force at the distance a or 90° is $= f$, and consequently by the theorem investigated in page 162, Vol. I, the time of a semivibration is $\sqrt{(a^3 \div 8f)}$ the same as is deduced from the more general expression.

that when the points of quiescence of the balance and auxiliary springs are coincident, the auxiliary spring commencing its action from the extremity B of the arc BO, (fig. 84, pl. 5.) continues to accelerate the balance till it arrives at the quiescent position O, at which point the action of the auxiliary spring entirely ceases; on this account it is plain that another auxiliary spring, equal and similar to the former, having also the point of quiescence coincident with O, must be applied to act by retardation on the balance while it describes the arc OE, in order that the times of describing the arcs BO and OE, as well as these arcs themselves, may be equal. According, therefore, to this disposition of the auxiliary springs, the balance will describe each semicircle of its vibrations precisely in the same manner as if it was impelled by one spiral spring only, the strength of which is equal to that of the balance and either auxiliary spring, when wound to the same tension. Suppose the balance to vibrate from B to E, and, for the sake of distinction, let the auxiliary spring which accelerates the balance from the extremity of the arc B, in the direction BO, be called u ; and let the other auxiliary * spring which retards the balance in the semicircle OE be denoted by v ; consequently, when the balance vibrates in the contrary direction from E to B, the auxiliary spring v will accelerate the balance from E to O, and the other auxiliary spring u will retard it from O to B. In respect, therefore, to the spring u , BO is the first semicircle, and OE is the latter semicircle of vibration; and on a similar principle in respect to the spring v , EO is the first semicircle, and OB is the latter semicircle of vibration.

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In

* The circular arcs which are drawn interior to the circumference of the balance in the figures 82, 83, 84, pl. 5, are intended to represent those portions of the balance's vibration in which the auxiliary springs respectively act.

The accelerative force of the auxiliary spring u , at the tension or distance from quiescence $Qd = a$ being nf , the force of the same spring at the tension or distance $QR = ad - x$, is

$$+ (nf \div a) (ad - x).$$

The force of the auxiliary spring v , at the tension $NE = a$ being nf , at the tension NR , the force of this spring acting by retardation will be

$$- (nf \div a) x.$$

Sum of these forces acting on the circumference of the balance, when it has described

the arc $NR = (f \div a) \times (d - x + 2dn - 2nx).$

Let u be the space through which a body falls freely from rest by the acceleration of gravity, to acquire the velocity of the circumference at R ; the principles of acceleration give this equation; $u = (f \div a) \times (dx - x\dot{x} + 2dnx - 2nx\dot{x})$; and taking the fluents so that when $x = 0$, u may be $= (f \div 2a) \times (b^2 - d^2 + nb^2 - 3nd^2 + 2nbd)$; $u = (f \div 2a) \times (b^2 - d^2 + nb^2 - 3nd^2 + 2nbd + (2d + 4nd)x - (1 + 2n)x^2)$; if t is put to represent the time of describing the arc NR , $t = \sqrt{(a \div 2lf)} \times \dot{x} \div \sqrt{(b^2 - d^2 + nb^2 - 3nd^2 + 2nbd + (2d + 4nd)x - (1 + 2n)x^2)}$, and taking the fluents so that when $x = 0$, $t = 0$, and making $x = d$, the time of describing the arc $NO = \sqrt{(a \div 2lf(1 + 2n))} \times$ a circular arc of which the sine $= \sqrt{(d^2(1 + 2n))} \div (b^2 + n(b + d)^2 - 2nd^2)$, or because $b + d = c$, $t = \sqrt{(a \div 2lf(1 + 2n))} \times$ a circular arc of which the sine $= \sqrt{(d^2(1 + 2n))} \div (b^2 + nc^2 - 2nd^2)$.

The result is, that when the points of quiescence of the auxiliary springs are situated in the latter
semiares

semiarcs of the respective vibrations, the time in which the balance describes the semiarc BO will be $\equiv \sqrt{2a \div lf(n+1)} \times$ a circular arc of which the sine is $\sqrt{((b-d)(n+1) \div (2b+2nc)) + (a \div 2lf(1+2n))}$ \times a circular arc of which the sine is $\sqrt{(d^2(1+2n) \div (b^2+nc^2-2nd^2))}$.

Corollary.—If the points of quiescence of the auxiliary springs are in the latter semiarcs of vibration, and the vibration commences from the point N (fig. 83, pl. 5.), in this case $d = b$, or $c = 2d$, and by the solution above, the time of a semivibration becomes $\sqrt{(a \div 2lf(1+2n))} \times$ a circular arc of which the sine is $\sqrt{(d^2(1+2n) \div (d^2(1+2n)))}$, or an arc of $90^\circ = p \div 2$: wherefore the time of a semivibration $\equiv \sqrt{(ap^2 \div 8lf(1+2n))}$. We observe, therefore, that whether the points of quiescence of the auxiliary springs are situated in the first or latter semiarcs of vibration, if the semiarc of vibration should be \equiv to the distance of the said points from the point of quiescence of the balance spring, the times of vibration will be the same whatever be the magnitude of that semiarc.

The balance of Mr. MUDGE's time-keeper describes the semiarc BO, by the joint action of two springs, i. e. the balance* spring, and an auxiliary spring; each spring is wound through the same arc BO, and comes to the same point of quiescence O (fig. 84, pl. 5.); consequently the action of the two springs is the same with that of a single spring of equal strength with both. In this case, the time of a semivibration through the arc BO will be obtained from a former solution; for referring to page 162, Vol. I. and making $OD = a$, and the force of the
balance

* A double spiral spring is applied in the balance of Mr. MUDGE's time-keeper, but as these two springs act as one spring, they are here considered as such.

balance spring and auxiliary spring at the distance from quiescence OD, $= f + nf = F$; the time of a semivibration is

$$= \sqrt{\frac{ap^2}{8lf(n+1)}} = \sqrt{\frac{ap^2}{8lF}}$$

(To be continued.)

ARTICLE XXII.

*Demonstrations to Dr. STEWART'S Propositions
proposed in ARTICLE XXI. Vol. I.*

PROP. XXI. THEO. XVIII. Fig. 164, 165,
166. Pl. 11.

Demonstrated by Mr. JOHN LOWRY.

Case I. WHEN the lines given by position are parallel to each other, fig. 164.

Let there be any number of right lines AL, BM, CN, DO, &c., given by position, and parallel to each other, and let $a, b, c, d, \&c.$ be given magnitudes, as many in number as there are right lines given by position; three right lines XQ, YR, ZS, may be found, that will be given by position, such, that if from any point P there be drawn perpendiculars PA, PB, PC, PD, &c. to all the right lines given by position, and likewise perpendiculars PX, PY, PZ, to the three lines found; the square of PA, together with the space to which the square of PB has the same ratio that a has to b , together with the space to which the square of PC has the same ratio that a has to c , and so on, will be

be equal to the space to which the sum of the squares of PX , PY , PZ , has the same ratio that thrice a has to the sum of a , b , c , &c.

From the point P draw $PABCD$, &c. perpendicular to the right lines given by position, and by Prop. X. find the point W for the points A , B , C , D , &c. and for the given magnitudes a , b , c , d , &c. In the right line PD take any point X , and divide XW in a , so that the space to which the rectangle XaW has the same ratio that thrice a has to twice the sum of a , b , c , &c. may be equal to the difference between the space to which the square of XW has the same ratio that thrice a has to twice the sum of a , b , c , &c. and the square of WA , together with the space to which the square of WB has the same ratio that a has to b , together with the space to which the square of WC has the same ratio that a has to c , and so on. Make $WY = Wa$, and $WZ = aX$, and through the three points X , Y , Z , draw three right lines XQ , YR , ZS , parallel to the right lines given by position, and they will be three such lines as are required.

It may be shewn as in Prop. XVII. Case I. that the space to which the sum of the squares of XW , YW , and ZW has the same ratio that thrice a has to the sum of a , b , c , &c. is equal to the square of AW , together with the space to which the square of BW has the same ratio that a has to b , together with the space to which the square of CW has the same ratio that a has to c , and so on.

But, by Prop. X. the square of EA , together with the space to which the square of PB has the same ratio that a has to b , together with the space to which the square of PC has the same ratio that a has to c , and so on, is equal to the square of AW , together with the space to which the square of BW has the same ratio that a has to b , together with the space to which

which the square of CW has the same ratio that a has to c , and so on, together with the space to which the square of PW has the same ratio that a has to the sum of $a, b, c, \&c.$ Therefore the square of PA , together with the space to which the square of PB has the same ratio that a has to b , together with the space to which the square of PC has the same ratio that a has to c , and so on, is equal to the space to which the sum of the squares of $XW, YW,$ and ZW , together with thrice the square of PW , has the same ratio that thrice a has to the sum of $a, b, c, \&c.$ But, since XW is equal to the sum of YW and ZW , it follows from Prop. IX. that the sum of the squares of $PX, PY,$ and PZ , is equal to the sum of the squares of $XW, YW,$ and ZW , together with thrice the square of PW ; therefore the square of PA , together with the space to which the square of PB has the same ratio that a has to b , together with the space to which the square of PC has the same ratio that a has to c , and so on, is equal to the space to which the sum of the squares of $PX, PY,$ and PZ , has the same ratio that thrice a has to the sum of $a, b, c, \&c.$

Case II. When the lines given by position intersect each other in a point, fig. 165.

Let there be any number of right lines $AB, AC, AD, AE, \&c.$ given by position, and intersecting each other in the point A , and let $a, b, c, d, \&c.$ be given magnitudes, as many in number as there are right lines given by position; three right lines AX, AY, AZ , may be found, that will be given by position, such, that if from any point P there be drawn perpendiculars $PB, PC, PD, PE, \&c.$ to all the right lines given by position, and likewise perpendiculars PX, PY, PZ , to the three lines found; the square of PB , together with the space to which the square of PC has the same ratio that a has to b , together

together with the space to which the square of PD has the same ratio that a has to c , and so on, is equal to the space to which the sum of the squares of PX, PY, PZ, has the same ratio that thrice a has to the sum of a, b, c , &c.

Draw AP, and upon it, as a diameter, let a circle be described, intersecting the lines given by position in B, C, D, E, &c. By Prop. X. find a point R for the points B, C, D, E, &c. and the given magnitudes a, b, c, d , &c. and by Prop. XIX. find two right lines AI, AK, for the point P, the right lines AB, AC, AD, AE, &c. given by position, and the given magnitudes a, b, c, d , &c. From any point Z, in the circumference of the circle, draw ZR, and continue it, so that RZ may be double of RS. Let O be the centre of the circle, and join OS, and let XSY be drawn perpendicular to SO, meeting the circle in X and Y. Join AX, AY, AZ, and they will be three such lines as are required.

By Prop. XIX. the square of PB, together with the space to which the square of PC has the same ratio that a has to b , together with the space to which the square of PD has the same ratio that a has to c , and so on, is equal to the space to which the sum of the squares of PI, PK, has the same ratio that twice a has to the sum of a, b, c , &c. And it may be shewn, in the same way as in Prop. XVII. Case 2, that twice the sum of the squares of PI, PX, is equal to the space to which the sum of the squares of PX, PY, and PZ, has the same ratio that 3 has to 2; therefore the square of PB, together with the space to which the square of PC has the same ratio that a has to b , together with the space to which the square of PD has the same ratio that a has to c , and so on, is equal to the space to which the sum of the squares of PX, PY, and PZ, has the same ratio that thrice a has to the sum of a, b, c , &c.

Case III. When the lines given by position are neither all parallel, nor intersecting each other in one point, fig. 166.

Let there be any number of right lines AB, BC, CD, AD, &c. given by position, that are neither all parallel, nor intersecting each other in one point, and let $a, b, c, d, \&c.$ be given magnitudes as many in number as there are right lines given by position; three right lines Ym, Zn, Kr may be found, that will be given by position, such, that if from any point X there be drawn perpendiculars XF, XE, XG, XH, &c. to all the right lines given by position, and likewise perpendiculars Xm, Xn, Xr, to the three lines found, the square of XE, together with the space to which the square of XF has the same ratio that a has to b , together with the space to which the square of XG has the same ratio that a has to c , and so on, is equal to the space to which the sum of the squares of Xm, Xn, and Xr, has the same ratio that thrice a has to the sum of $a, b, c, \&c.$

Find by Prop. XX. two right lines PV, PW, for the point X, the right lines AB, BC, CD, AD, &c. given by position, and the given magnitudes $a, b, c, d, \&c.$ From the point P draw PM, PO, PR, PQ, &c. perpendicular, and PL, PN, PS, PU, &c. parallel to AB, BC, CD, AD, &c. and let PL, PN, PS, PU, &c. meet the perpendiculars XE, XF, XG, XH, &c. drawn from the point X to the right lines AB, BC, CD, AD, &c. in L, N, S, U, &c. Find, by the preceding case, three right lines Pa, Pb, Pc, for the point X, the right lines PL, PN, PS, PU, &c. given by position, and the given magnitudes $a, b, c, d, \&c.$ and from any point, as c , in Pc draw Cq parallel to Pa meeting Pb in q . Then, by Prop. XVI. find the intersection w of two right lines, for the point X and the right lines Pq, Pc, Cq; and from w draw wf, wg, wh, perpendicular to Cq, Pc, Pq. Let

Let a square be found to which the square of PM together with the space to which the square of PO has the same ratio that a has to b , together with the space to which the square of PR has the same ratio that a has to c , and so on, has the same ratio that the sum of $a, b, c, d, \&c.$ has to thrice a . Divide this square into three others, whose sides shall have the same ratio to each other as the three lines wf, wg, wh . Draw PK, PY, PZ, parallel to wf, wg, wh , and thereon take PK, PY, PZ, equal to the sides of the three squares just found, that is, on lines which are parallel to those to which the sides of the squares are respectively proportional. Then draw Ym, Zn, Kr , perpendiculars to PY, PZ, PK, and they will be three such lines as are required.

Draw $Xm, Xn, Xr, Xa, Xb, Xc, XV, XW$, respectively parallel to $Ym, Zn, Kr, Pa, Pb, Pc, PV, PW$.

It appears, in the same way as in Prop. XVII. that the square of XE, together with the space to which the square of XF has the same ratio that a has to b , together with the space to which the square of XG has the same ratio that a has to c , and so on, is equal to the square of PM, together with the space to which the square of PO has the same ratio that a has to b , together with the space to which the square of PR has the same ratio that a has to c , and so on, together with the square of XL, together with the space to which the square of XN has the same ratio that a has to b , together with the space to which the square of XS has the same ratio that a has to c , and so on. Also, that the sum of the squares of Xm, Xn, Xr , is equal to the sum of the squares of Xa, Xb, Xc , together with the sum of the squares of PY, PZ, PK. Moreover, by the preceding case, the square of XL, together with the space to which the square of XN has the same ratio that a has to b , together with the space to which the square of XS

has the same ratio that a has to c , and so on, is equal to the space to which the sum of the squares of Xa , Xb , Xc , has the same ratio that thrice a has to the sum of a , b , c , &c. and, by construction, the square of PM , together with the space to which the square of PO has the same ratio that a has to b , together with the space to which the square of PR has the same ratio that a has to c , and so on, is equal to which the sum of the squares of PY , PZ , PK , has the same ratio that thrice a has to the sum of a , b , c , &c. Therefore the square of PM , together with the space to which the square of PO has the same ratio that a has to b , together with the space to which the square of PR has the same ratio that a has to c , and so on, together with the square of XL , together with the space to which the square of XN has the same ratio that a has to b , together with the space to which the square of XS has the same ratio that a has to c , and so on, is equal to the space to which the sum of the squares of Xa , Xb , Xc , has the same ratio that thrice a has to the sum of a , b , c , &c. together with the space to which the sum of the squares of PY , PZ , PK , has the same ratio that thrice a has to the sum of a , b , c , &c. Therefore the square of XE , together with the space to which the square XF has the same ratio that a has to b , together with the space to which the square of XG has the same ratio that a has to c , and so on, is equal to the space to which the sum of the squares of Xm , Xn , Xr , has the same ratio that thrice a has to the sum of a , b , c , &c.

The same demonstrated by Mr. Swale, Fig. 205, Pl. 15.

Let there be any number of right lines, AB , CD , EF , &c. given by position, and let a , b , c , &c. be given magnitudes; as many in number as there are right lines given by position; three right lines QR ,
 ST ,

ST, UV, may be found that will be given by position, such, that if from any point P there be drawn perpendiculars PA, PC, PE, &c. to all the right lines given by position, and likewise there be drawn perpendiculars PS, PQ, PU, to the three lines found, the square of PA, together with the space to which the square of PC has the same ratio that a has to b , together with the space to which the square of PE has the same ratio that a has to c , and so on, will be equal to the space to which the sum of the squares of PS, PU, PQ, has the same ratio that thrice a has to the sum of a , b , c , &c.

Suppose three right lines AB, CD, EF, given by position. From any point P, demit the \perp s PA, PC, PE; join AC, and take AG : GC as $b : a$; join EG, and take GH : HE as C : the sum of a , b ; join PG, PH; erect the \perp s GI, HK, to meet semicircles described upon AC, EG, in I, K; take GL : GC as EK : EG, and let a semicircle described on AL to meet GI in N; then in PG, PH, and any other line PU, taking PQ, PS, PU, equal to the sides of the squares $3GN^2$, $3PH^2$, $3HK^2$, respectively, and drawing QR, ST, UV, \perp thereto, they will be three such lines as are required.

By Prop. XX. the square of PA, together with the space to which the square of PC has the same ratio that a has to b , together with the space to which the square of PE has the same ratio that a has to c , is equal to the space to which the sum of the squares PH, KH, has the same ratio that a has to the sum of a , b , c , together with the space to which the square of GI has the same ratio that a has to the sum of a , b . But the square of GN is to the square of GI, as GL to GC, or, as EH to EG, or, as the sum of a , b , to the sum of a , b , c ; therefore the space to which the square of GI has the same ratio that a has to the sum of a , b , is equal to the space to which the

square of GN has the same ratio that a has to the sum of a, b, c . Wherefore, the square of PA, together with the space to which the square of PC has the same ratio that a has to b , together with the space to which the square of PE has the same ratio that a has to c , is equal to the space to which the sum of the squares of PH, HK, GN, has the same ratio that a has to the sum of a, b, c , that is, equal to the space to which thrice the sum of the squares of PH, HK, GN, has the same ratio that thrice a has to the sum of a, b, c ; that is, equal to the space to which thrice the sum of the squares of PH, HK, GN, has the same ratio that thrice a has to the sum of a, b, c ; that is, equal to the space to which the sum of the squares of PS, PU, PQ has the same ratio that thrice a has to the sum of a, b, c .

Cor. Let there be any number of right lines given by position; three right lines may be found that will be given by position, such, that if from any point there be drawn right lines in given angles to all the right lines given by position, and likewise there be drawn perpendiculars to the three lines found, the sum of the squares of the lines drawn in given angles to the right lines given by position, will be equal to the space to which the sum of the squares of the perpendiculars, drawn to the three lines found, has a given ratio.

PROP. E. THEO.

Added by Mr. Lowry.

In the right line AB (fig. 221, pl. 15.) let there be three points A, D, B; then I say, if D be between A and B, that the cube on AB will be equal to the sum of the cubes on AD, DB, together with thrice the

the solid whose base is the rectangle ADB , and altitude AB : But if B be between A and D , then I say, that the cube on AB , together with thrice the solid whose base is the rectangle ADB , and altitude AB , will be equal to the difference of the cubes on AD , DB .

Case I. When D is between A and B .

Let the cube $AFKH$, whose side is AB , be intersected by the plane $DqLN$ parallel to the sides $AEIK$, $BFHG$, of the cube and passing through the point D . Let the cubes $AWbc$, $DPUS$, whose sides are AD , DB , be described, and the lines drawn as in the figure. Then the solid $EDIN$ is equal to the sum of the solids $EWIn$, $XDmN$. But the solid $XDmN$ is equal to the cube on AD , together with the solid $admN$; therefore the solid $EDIN$ is equal to the sum of the solids $EWIn$, $admN$, together with the cube on AD . But AX and Ab are each equal to AD , and AE is equal to AK , therefore EX and bK are each equal to DB ; wherefore, the bases EW , Kd , of the solids $EWIn$, $admN$, are each equal to the rectangle ADB , and their altitudes AK and ab , to AB and AD respectively. Therefore the sum of the solids $EWIn$, $admN$, is equal to the solid whose base is the rectangle ADB , and altitude the sum of AB , AD , therefore the solid $EDIN$, is equal to the cube on AD , together with the solid whose base is the rectangle ADB , and altitude the sum of AB , AD .

In the same way it may be shewn, that the solid $qBLG$ is equal to the cube on DB , together with the solid whose base is the rectangle ADB and altitude AB , BD . Therefore the sum of the solids $EDIN$, $qBLG$, that is, the cube on AB , is equal to the sum of the cubes on AD , DB , together with the solid whose base is the rectangle ADB and altitude the sum of AB , AD , together with the solid whose base is the

the rectangle ADB and altitude AB, BD. But the sum of AB, AD, together with the sum of AB, BD, is equal to thrice AB, therefore, the cube on AB is equal to the sum of the cubes on AD, DF, together with thrice the solid whose base is the rectangle ADB, and altitude AB.

Case II. When B is between A and D. Fig. 222, pl. 15.

It has been shewn, that the cube on AD is equal to the sum of the cubes on AB, BD, together with thrice the solid whose base is the rectangle ABD, and altitude AD. But the solid whose base is the rectangle ABD, and altitude AD is evidently equal to the solid whose base is the rectangle ADB, and altitude AB, therefore the cube on AD is equal to the sum of the cubes on AB, DB, together with thrice the solid whose base is the rectangle ADB, and altitude AB. Wherefore, the cube on AB, together with thrice the solid whose base is the rectangle ADB, and altitude AB is equal to the difference of the cubes on AD and DB.

Cor. The cube on any right line is equal to eight times the cube on half that line.

PROP. F. THEO.

Added by Mr. LOWRY.

In the right line AB, let there be four points A, B, C, D, and let the point C, be equidistant from A and B: then, if D be between A and B, the sum of the cubes on AD, DB, will be equal to twice the cube AC or CB, together with thrice the solid whose base is the square of DC and altitude AB: but if B be between A and D, the difference of the cubes on AD, DB, will be equal to twice the cube on AC or CB, together with thrice the solid whose base is the square of CD, and altitude AB.

Case

Case I. When D is between A and B. Fig. 221, pl. 15.

By Prop. E. the sum of the cubes on AD, DB, together with thrice the solid whose base is the rectangle ADB, and altitude AB, is equal to the cube on AB, that is, (*by cor. ibid.*) equal to eight times the cube on AC. But the rectangle ADB, is equal to the excess of the square on BC above the square on DC, therefore the solid whose base is the rectangle ADB, and altitude AB, is equal to the solid whose base is the excess of the square on BC above the square on DC, and altitude AB, that is, equal to the excess of twice the cube on BC above the solid whose base is the square of DC, and altitude AB, therefore, thrice the solid whose base is the rectangle ADB, and altitude AB is equal to the excess of six times the cube on BC above thrice the solid whose base is the square of DC, and altitude AB. Wherefore the sum of the cubes on AD, DB, is equal to the cube on AC, or BC, together with thrice the solid whose base is the square of DC, and altitude AB.

Case II. When B is between A and D.

This follows in the same manner from *Case 2*, of Prop. E.

PROP. XXII. THEO. XIX. Fig. 268, 269. Pl. 17.

Demonstrated by Mr. Lowry.

Case I. When the number of the sides of the figure circumscribed about the circle is even, fig. 268.

Let ABCDEF, &c. be any regular figure of an even number of sides circumscribed about a circle, and from any point G, in the circumference, let there be drawn GH, GK, GL, GM, GN, GO, &c. perpendicular to the sides of the figure, and let *a* be the centre of the circle, and join Ga; twice the sum of
of

of the cubes on $GH, GK, GL, GM, GN, GO,$ &c., will be equal to five times the multiple of the cube on the semidiameter aG , of the circle, by the number of the sides of the figure.

Let the circumscribing figure touch the circle in $P, Q, R, S, T, V,$ &c. and join $aP, aQ, aR,$ &c. and draw $GX, GY, GZ,$ &c. perpendicular to $aP, aQ, aR,$ &c.

Because the number of the sides of the circumscribing figure is even, it is plain, that $aP, aQ, aR,$ &c. will pass through the opposite points of contact, that is, through the points $S, T, V,$ &c. and that $PS, QT, RV,$ &c. are bisected in a ; therefore, (Prop. F.) the sum of the cubes on $GH, GM,$ that is, the sum of the cubes on $PX, XS,$ is equal to twice the cube on aP , together with six times the solid whose base is the square of aX and altitude aP . In the same way it is shewn, that the sum of the cubes on $GK, GN,$ is equal to twice the cube on aP , together with six times the solid whose base is the square of aY , and altitude aP ; and likewise, that the sum of the cubes on $GO, GL,$ is equal to twice the cube on aP , together with six times the solid whose base is the square of aZ and altitude aP ; and so on. Therefore, the sum of the cubes on $GH, GK, GL, GM, GN, GO,$ &c. is equal to the multiple of the cube on aP by the number of the sides of the figure, together with six times the multiple, by the same number, of the solid whose base is the sum of the squares of $aX, aY, aZ,$ &c. and altitude aP .

Again, because the angles $GXA, GYA, GZA,$ &c. are right angles, the points $X, Y, Z,$ &c. will be in the circumference of the circle whose diameter is Ga ; and because the circle passes through the point a , the circumference will be divided into equal parts in the points $X, Y, Z,$ &c. as many in number as there are right lines $aP, aQ, aR,$ &c. (Lem. II.).

There-

Therefore (Prop. IV.) the sum of the squares of aX , aY , aZ , &c. is equal to twice the multiple of the square of half aG , by the number of the lines aP , aQ , aR , &c. that is, equal to the multiple of the square of half aG , by the number of the sides of the circumscribing figure. Therefore, twelve times the multiple, by the number of the sides of the circumscribing figure, of the solid whose base is the sum of the squares of aX , aY , aZ , &c. and altitude aP , is equal to thrice the multiple, by the same number, of the cube on aP : and therefore twice the sum of the cubes on GH , GK , GL , GM , GN , GO , &c. is equal to five times the multiple of the cube on aP , by the number of the sides of the figure.

Case II. When the number of the sides of the figure circumscribed about the circle is odd, fig. 269.

Let $ABCDE$, &c. be any regular figure of an odd number of sides circumscribed about a circle, and from any point in the circumference, let there be drawn PH , PK , PL , PM , PN , &c. perpendicular to the sides of the figure, and let a be the centre of the circle, and join Pa ; twice the sum of the cubes on PH , PK , PL , PM , PN , &c. will be equal to five times the multiple of the cube on the semidiameter, aP , of the circle by the number of the sides of the figure,

Let the circumscribing figure touch the circle in the points Q , R , S , T , U , &c. and join aQ , aR , aS , aT , aU , &c.; join also PQ , PR , PS , PT , PU , &c. and draw PV , PW , PX , PY , PZ , &c. perpendicular to aQ , aR , aS , aT , aU , &c. The cube on PH or QV is equal to the solid whose base is the difference of the squares of PQ , PV , and altitude QV : but the square of PV is equal to the difference of the squares of aP , aV , and the square of PQ is equal to twice the rectangle aPH (Lem. I.); therefore the difference of the squares of PQ and PV is equal

equal to the difference of twice the rectangle aPH , together with the square of aV , and the square of aP ; therefore the cube on PH is equal to the solid whose base is the difference of twice the rectangle aPH , together with the square of aV , and the square of aP , and altitude PH . The same way it is shewn, that the cube on PK is equal to the solid whose base is the difference of twice the rectangle aPK , together with the square of aW , and the square of aP , and altitude PK ; that the cube on PL is equal to the solid whose base is the difference of twice the rectangle aPL , together with the square of aX , and the square of aP , and altitude PL ; that the cube on PM is equal to the solid whose base is the difference of twice the rectangle aPM , together with the square of aY , and the square of aP , and altitude PM ; that the cube on PN is equal to the solid whose base is the difference of twice the rectangle aPN , together with the square of aZ , and the square of aP , and altitude PN ; and so on. Now, the solid whose base is twice the rectangle aPH , and altitude PH , together with the solid whose base is twice the rectangle aPK , and altitude PK , together with the solid whose base is twice the rectangle aPL , and altitude PL , together with the solid whose base is twice the rectangle aPM , and altitude PM , together with the solid whose base is twice the rectangle aPN , and altitude PN , and so on, is equal to the solid whose base is twice the sum of the squares of PH , PK , PL , PM , PN , &c. and altitude aP . But the sum of the squares of PH , PK , PL , PM , PN , &c. is equal to thrice the multiple of the square of aP by the number of the sides of the figure (Prop. V.) Therefore the solid whose base is twice the sum of the squares of PH , PK , PL , PM , PN , &c. and altitude aP , is equal to thrice the multiple of the cube on aP , by the number of the sides of the figure.

Again,

Again, the solid whose base is the square of aP , and altitude the sum of PH , PK , PL , PM , PN , &c. is equal to the multiple of the cube on aP , by the number of the sides of the figure. Therefore twice the sum of the cubes on PH , PK , PL , PM , PN , &c. is equal to four times the multiple of the cube on aP , by the number of the sides of the figure, together with the solid whose base is twice the square of aV , and altitude PH , together with the solid whose base is twice the square of aW , and altitude PK , together with the solid whose base is twice the square of aX , and altitude PL , together with the solid whose base is twice the square of aY , and altitude PM , together with the solid whose base is twice the square of aZ , and altitude PN , and so on. Now, it is easily shewn, that the square of aV , together with the space to which the square aW has the same ratio that PH has to PK , together with the space to which the square of aX has the same ratio that PH has to PL , together with the space to which the square of aY has the same ratio that PH has to PM , together with the space to which the square of aZ has the same ratio that PH has to PN , and so on, is equal to the space to which the square of aP has the same ratio that twice PH has to the sum of PH , PK , PL , PM , PN , &c. that is, the same ratio that twice PH has to the multiple of aP by the number of the sides of the figure. Therefore the solid whose base is twice the square of aV , and altitude PH , together with the solid whose base is twice the square of aW , and altitude PK , and so on, is equal to the multiple of the cube on aP , by the number of the sides of the figure. And, therefore, twice the sum of the cubes on PH , PK , PL , PM , PN , &c. is equal to five times the multiple of the cube on aP , by the number of the sides of the figure.

*The same demonstrated by Mr. Swale, Fig. 270, 271.
Pl. 17.*

Case I. When the number of the sides is even, fig. 270.

Let ABCDEFA, be a regular figure of six sides circumscribed about the circle GHIKLM, whose centre is O. From any point P, in the circumference, let there be drawn PQ, PR, PS, PT, PV, PW, perpendicular to the sides of the figure; twice the sum of the cubes on the perpendiculars PQ, PR, PS, PT, PV, PW, will be equal to five times the multiple of the cube on the semidiameter, OP, of the circle, by the number AB, BC, CD, DE, EF, FA, of the sides of the figure.

Draw OG, OH, OI, OK, OL, OM, to the points of contact; also join OP, and demit, upon OK, OI, OH, the perpendiculars Pd , Pm , Pn . Because the figure has an even number of sides, it is evident, that AB, DE, BC, EF; and CD, FA, will be respectively parallel to each other, and consequently, GOK, HOL, LOM, joining the points of contact, will be straight lines, and each equal to the diameter.

Also, $PQ = dG$; $PR = nH$; $PS = mI$, $PT = dK$; $PV = nH$; $PW = mM$; and consequently, twice the sum of the cubes on the \perp s PQ, PR, PS, FT, PV, PW is equal to twice the sum of the cubes on dG , dK , nH , nL , mI , mM .

The sum of the cubes on dG , dK , is equal to the solid whose base is the difference between the sum of the squares dG , dK , and the rectangle GdK , and altitude GK, that is, equal to the solid whose base is the difference between the square of GK, and thrice the rectangle GdK and altitude GK, that is, equal to the solid whose base is the difference between the square of GK, and thrice the square of dP , and
altitude

altitude GK; therefore twice the sum of the cubes on dG , DK , is equal to the solid whose base is the difference between the square of GK, and thrice the square of dP , and altitude twice GK. In the same way it is shewn, that twice the sum of the cubes on nH , nL , is equal to the solid whose base is the difference between the square of HL, and thrice the square of nP , and altitude twice HL; that is twice the sum of the cubes on mI , Mm , is equal to the solid whose base is the difference between the square of IM and thrice the square of mP , and altitude twice IM. Therefore twice the sum of the cubes on dG , dK , nK , nL , mI , mM , is equal to thrice the solid whose base is the difference between the square of GK, and the sum of the squares of dP , nP , mP , and altitude twice GK, that is, equal to the solid whose base is the difference between twice the square of GK, and twice the sum of the squares of dP , nP , mP , and altitude thrice GK. But twice the square of GK is equal to eight times the square of OP; and since the angles about the point O are equal, twice the sum of the squares of dP , nP , mP , is equal to thrice the square of OP (Prop. XIV.); also thrice the diameter GK is equal to six times the radius OP. Therefore, twice the sum of the cubes on dG , dK , nH , nL , mI , mM , is equal to the solid whose base is the difference between eight times the square of OP, and altitude six times OP, that is, equal to the solid whose base is five times the square of OP, and altitude six times OP, that is, equal to five times the multiple of the cube on OP, by the number six. And, therefore, twice the sum of the cubes on PQ, PR, PS, PT, PV, PW, is equal to five times the multiple of the cube on the semidiameter, OP, of the circle, by the number of the sides of the figure.

Case II. When the number of the sides is odd, (fig. 271.)

Let ABCDEA, be a regular figure of five sides circumscribed about the circle LNQRS, whose centre is O. From any point P, in the circumference, let there be drawn PG, PH, PI, PK, PF, perpendicular to the sides of the figure; twice the sum of the cubes on the perpendiculars PG, PH, PI, PK, PF, will be equal to five times the multiple of the cube on the semidiameter, OP, of the circle, by the number AB, BC, CD, DE, EA, of the sides of the figure.

Join the centre O, and the points of contact L, N, Q, R, S; join also P and the points L, N, Q, R, S, and upon OL, ON, OQ, OR, OS, demit the perpendiculars PT, PV, PW, PX, PM. Then will $PG = TL$, $PH = VN$, $PI = WQ$, $PK = RX$, $PF = SM$; and $PO = LO = NO = QO = RO = SO$, and twice the sum of the cubes on PG, PH, PI, PK, PF, is equal to twice the sum of the cubes on TL, VN, WQ, XR, MS.

Now, twice the sum of the cubes on LT, LO, is equal to the solid whose base is the difference between twice the sum of the squares of LT, LO, and twice the rectangle TLO, and altitude the sum of LT, LO; twice the sum of the cubes on NV, LO is equal to the solid whose base is the difference between twice the sum of the squares of NV, LO, and twice the rectangle VNO, and altitude the sum of NV, LO; twice the sum of the cubes on QW, LO, is equal to the solid whose base is the difference between twice the sum of the squares of QW, LO, and twice the rectangle WQO, and altitude the sum of WQ, LO; twice the sum of the cubes on RX, LO, is equal to the solid whose base is the difference between

between twice the sum of the squares of RX , LO , and twice the rectangle XRO , and altitude the sum of XR , LO ; and twice the sum of the cubes on SM , LO , is equal to the solid whose base is the difference between twice the sum of the squares of SM , LO , and twice the rectangle MSO , and altitude the sum of SM , LO . Therefore, twice the sum of the cubes on LT , NV , QW , RX , SM , together with ten times the cube on LO , is equal to the solid whose base is the difference between twice the sum of the squares of LT , LO , and twice the rectangle TLO , and altitude the sum of LT , LO , together with the solid whose base is the difference between twice the sum of the squares of NV , LO , and twice the rectangle VNO , and altitude the sum of NV , LO , together with the solid whose base is the difference between twice the sum of the squares of QW , LO , and twice the rectangle WQO , and altitude the sum of WQ , LO , together with the solid whose base is the difference between twice the sum of the squares of RX , LO , and twice the rectangle XRO , and altitude the sum of XR , LO , together with the solid whose base is the difference between twice the sum of the squares of SM , LO , and twice the rectangle MSO , and altitude the sum of SM , LO , that is, equal to the solid whose base is the square of PL , together with twice the square of OT , and altitude the sum of LT , LO , together with the solid whose base is the square of NP , together with twice the square of VO , and altitude the sum of NV , LO , together with the solid whose base is the square of QP , together with twice the square of WO , and altitude the sum of QW , LO ; together with the solid whose base is the square of RP , together with twice the square of XO , and altitude the sum of RX , LO ; together with the solid whose base is the square of SP , together with twice the square of

O 3

MO,

MO, and altitude the sum of SM, LO, that is, equal to the solid whose base is the sum of the squares of PL, PN, PQ, PR, PS, together with twice the sum of the squares of OT, OV, OW, OX, OM, and altitude LO, together with the solid whose base is the square of PL, together with twice the square of OT, and altitude LT, together with the solid whose base is the square of PN, together with twice the square of OV, and altitude NV, together with the solid whose base is the square of PQ, together with twice the square of OW, and altitude QW, together with the solid whose base is the square of PR, together with twice the square of OX, and altitude RX, together with the solid whose base is the square of PS, together with twice the square of OM, and altitude SM, that is, equal to the solid whose base is the difference between the sum of the squares of PL, PN, PQ, PR, PS, together with ten times the square of LO, and twice the sum of the squares of PT, PV, PW, PX, PM, and altitude LO, together with the solid whose base is twice the square of LO, and altitude the sum of LT, NV, QW, RX, SM, together with the difference between the solid whose base is twice the sum of the squares of TL, NV, QW, RX, SM, and altitude LO, and the sum of the solids whose bases are twice the squares of PT, PV, PW, PX, PM, and altitudes LT, NV, QW, RX, SM, respectively. But (Prop. IV.) the sum of the squares of PL, PN, PQ, PR, PS, is equal to ten times the square of LO; and (Prop. XIV.) twice the sum of the squares of PT, PV, PW, PX, PM, is equal to five times the square of LO; also (Prop. III.) the sum of LT, NV, QW, RX, SM, is equal to five times LO; and (Prop. V. Cor. I.) twice the sum of the squares of TL, NV, QW, RX, SM, is equal to fifteen times the square of LO. Therefore, the solid whose base is the sum of the squares of PL, PN, PQ, PR, PS, and altitude

LO,

LO, is equal to ten times the cube on LO; the solid whose base is ten times the square of LO, and altitude LO, is equal to ten times the cube on LO; the solid whose base is twice the sum of the squares of PT, PV, PW, PX, PM, and altitude LO, is equal to five times the cube on LO; the solid whose base is twice the square of LO, and altitude the sum of LT, NV, QW, RX, SM, is equal to ten times the cube on LO; the solid whose base is twice the sum of the squares of TL, NV, QW, RX, SM, and altitude LO, is equal to fifteen times the cube on LO; and the solid whose base is twice the square of PT, and altitude LT, together with the solid whose base is twice the square of PV, and altitude NV, together with the solid whose base is twice the square of PW, and altitude QW, together with the solid whose base is twice the square of PX, and altitude RX, together with the solid whose base is twice the square of PM, and altitude SM, is equal to five times the cube on LO. And, therefore, twice the sum of the cubes on LT, NV, QW, RX, SM, that is, twice the sum of the cubes on PG, PH, PI, PK, PF, together with ten times the cube on LO, is equal to the difference between ten times the cube on LO, together with ten times the cube on LO, together with ten times the cube on LO, together with fifteen times the cube on LO, and five times the cube on LO, that is, equal to the difference between forty-five times the cube on LO, and ten times the cube on LO, that is, equal to thirty-five times the cube on LO. Therefore, twice the sum of the cubes on the perpendiculars PG, PH, PI, PK, PF, is equal to the difference between thirty-five times the cube on LO, and ten times the cube on LO, that is, equal to twenty-five times the cube on LO, that is, equal to five times the multiple of the cube on the semidiameter, OP, of the circle, by the number of the sides of the figure.

ARTICLE XXIII.

*Demonstrations to Lawson's Propositions proposed in
ARTICLE III.*

PROP. XXVIII. Fig. 272, 273, Pl. 17.

Demonstrated by Mr. Colin Campbell.

JOIN CH, CK, and draw the radii OH, OK.
 Since, the rect. $ACB =$ the rect. ECF ,
 CFE , or $ECF + CF^2 = ACB + CF^2 = KFG$ (Prop. XI.);
 and FEC , or $ECF + CE^2 = ACB + CE^2 = HEG$ (Prop. XI.);
 wherof. $FK : FE :: FC : FG$, and $EH : EF :: EC : EG$;
 consequently $\angle FCK = EGF = ECH$.
 Hence $\angle HOK + HCK = HCK + 2FCK = 2$ right \angle s,
 therefore the points H, O, K, C, are in a circle;
 wherfore $CLO = HLK = ALB$.
 Hence $AL : LB :: AC : CB$ (Conv. Prop. I).
Q. E. D.

The same by Mr. Harris.

ANALYSIS.

Join CK and let it meet the circle again in I; join
 also CH, IH.—Since $AL : LB :: AC : CB$,
 by Prop. VII. HI. is \perp to AB, and conseq. \parallel to CD;
 therof. the $\angle FCK = HIK = HGK$ or EGF ;
 but, by Prop. IV. $\angle FCK = ECH$; hence, $\angle ECH = EGF$;
 consequently the points C, H, K, G, are in a circle;
 therefore the rect. $CEF =$ the rect. HEG ;
 but, by Prop. XI. $ACB + CE^2 = HEG$;
 therefore. CEF , or $ECF + CE^2 = ACB + CE^2$;
 and therefore the rect. $ECF =$ the rect. ACB ,
Q. Q. V.

The

The same by Mr. Lowry.

Draw CK to meet the circle in I and join HI.
 By hypothesis, $ECF = ACB$; add CF^2 to each,
 then $CFE = ECF^2 + CF^2 = ACB + CF^2 = KFG$ (Prop. XI.);
 therefore the Δ s EFG, CFK, are equiangular;
 hence, $\angle FCK = HGK = HIC$;
 theref. HI is parallel to EF, and perpend. to AB,
 consequently, by Prop. VII. $AL : LB :: AC : CB$.
Q. E. D.

PROP. XXIX. Fig. 274, 275. Pl. 17.

*Demonstrated by Messrs. Campbell, Harris, and
 Lowry.*

ANALYSIS, by Mr. Harris.

Since FG is parallel to AE by hypothesis,
 the $\angle AEC = EFG = CDG$ or ADE;
 therefore the Δ s AEC, ADE are equiangular,
 wherefore $AD : AE :: AE : AC$;
 hence $AE^2 = DAC = AB$; $\therefore AE = AB$.
Q. Q. V.

SYNTHESIS, by Messrs. Campbell and Lowry.

Because $AE = AB$, $AE^2 = AB^2 = CAD$,
 therefore $DA : AE :: AE : AC$;
 wherefore the Δ s ADE, ACE, are equiangular;
 hence, the $\angle AEC = ADE = EFG$.
 and FG is parallel to AE. Q. E. D.

PROP.

PROP. XXX. Fig. 276, 277, Pl. 17.

Demonstrated by Messrs. Campbell, Harris, and Lowry.

ANALYSIS, by Mr. Harris.

Because GH is parallel to AB, by hypothesis,
the $\angle AEC = CGH = CDH$;
wherefore the points E, C, D, F, are in a circle;
hence, rect. EAF = rect. CAD = AB^2 . Q. Q. V.

SYNTHESIS, by Messrs. Campbell and Lowry.

By Analysis, $EAF = AB^2 = CAD$;
therefore $AC : AF :: AE : AD$;
wherefore the Δ s ACE, ADF are equiangular,
therefore, $\angle AEC = ADF = EGH$;
consequently GH is parallel to AB. Q. E. D.

ARTICLE XXIV.

To the Editor of the Mathematical Repository.

SIR,

IN almost every book of fluxions, methods have been pointed out for finding the fluent of a fluxion

of this form, viz. $\frac{x^m}{x^n - px^{n-1} + qx^{n-2} - \&c.}$ where

all the roots of the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$,
are supposed to be possible. The following manner
of investigating this fluent is different from any I have
seen,

feen, and if you think that such a trifle deserves a place in your REPOSITORY, amongst the many valuable papers to be found there, by inserting it you will oblige,

Sir, your humble Servant,

A. B.

Required the fluent of $\frac{x^m \dot{x}}{x^n - px^{n-1} + qx^{n-2} - \&c.}$?

Let the roots of the equation $x^n - px^{n-1} + qx^{n-2} - \&c. = 0$, be equal to $a, b, c, d, \&c.$ and suppose

$$\frac{x^m \dot{x}}{x^n - px^{n-1} + qx^{n-2} - \&c.} = K \times \frac{x^m \dot{x}}{x-a} + L \times \frac{x^m \dot{x}}{x-b}$$

+ M $\times \frac{x^m \dot{x}}{x-c}$ + &c. then by division we shall have

$$\frac{x^m \dot{x}}{x^n - px^{n-1} + qx^{n-2} - \&c.} = \left\{ \begin{array}{l} +K \times \left(x^{m-1} \dot{x} + ax^{m-2} \dot{x} + a^2 x^{m-3} \dot{x} \dots + \frac{a^m \dot{x}}{x-a} \right) \\ +L \times \left(x^{m-1} \dot{x} + bx^{m-2} \dot{x} + b^2 x^{m-3} \dot{x} \dots + \frac{b^m \dot{x}}{x-b} \right) \\ +M \times \left(x^{m-1} \dot{x} + cx^{m-2} \dot{x} + c^2 x^{m-3} \dot{x} \dots + \frac{c^m \dot{x}}{x-c} \right) \\ + \quad \quad \quad \&c. \quad \quad \quad \&c. \end{array} \right.$$

$$\text{But } \frac{K}{x-a} + \frac{L}{x-b} + \frac{M}{x-c} + \frac{N}{x-d} + \&c. = \frac{1}{x^n - px^{n-1} + qx^{n-2} - \&c.},$$

therefore by reducing the fractions to a common denominator, we have Kx

Kx^{n-1}	$\begin{array}{l} -b \\ -c \\ -d \times Kx^{n-2} \\ -e \\ -\&c. \end{array}$	$\begin{array}{l} +bc \\ +bd \\ +be \\ +cd \times Kx^{n-3} \\ +ce \\ +de \\ +\&c. \end{array}$	$\begin{array}{l} -bcd \\ -bce \\ -bde \times Kx^{n-4} \dots \pm bcde \&c. \times K \\ -cde \\ -\&c. \end{array}$
$+Lx^{n-1}$	$\begin{array}{l} -a \\ -c \\ -d \times Lx^{n-2} \\ -e \\ -\&c. \end{array}$	$\begin{array}{l} +ac \\ +ad \\ +ae \\ +cd \times Lx^{n-3} \\ +ce \\ +de \\ +\&c. \end{array}$	$\begin{array}{l} -acd \\ -ade \\ -ace \times Lx^{n-4} \dots \pm acde \&c. \times L \\ -cde \\ -\&c. \end{array}$
$-Mx^{n-1}$	$\begin{array}{l} -a \\ -b \\ -d \times Mx^{n-2} \\ -e \\ -\&c. \end{array}$	$\begin{array}{l} +ab \\ +ad \\ +ae \\ +bd \times Mx^{n-3} \\ +be \\ +de \\ +\&c. \end{array}$	$\begin{array}{l} -abd \\ -ade \\ -abe \times Mx^{n-4} \dots \pm abde \&c. \times M \\ -bde \\ -\&c. \end{array}$
$+Nx^{n-1}$	$\begin{array}{l} -a \\ -b \\ -c \times Nx^{n-2} \\ -e \\ -\&c. \end{array}$	$\begin{array}{l} +ab \\ +ac \\ +ae \\ +bc \times Nx^{n-3} \\ +be \\ +ce \\ +\&c. \end{array}$	$\begin{array}{l} -abc \\ -abe \\ -ace \times Nx^{n-4} \dots \pm abce \&c. \times N \\ -bce \\ -\&c. \end{array}$
$+Px^{n-1}$	$\begin{array}{l} -a \\ -b \\ -c \times Px^{n-2} \\ -d \\ -\&c. \end{array}$	$\begin{array}{l} +ab \\ +ac \\ +ad \\ +bc \times Px^{n-3} \\ +bd \\ +cd \\ +\&c. \end{array}$	$\begin{array}{l} -abc \\ -abd \\ -acd \times Px^{n-4} \dots \pm abcd \&c. \times P \\ -bcd \\ -\&c. \end{array}$

— 1

It is evident

It is evident, from the last equation that
 $K + L + M + N + P + \&c. = 0$, and likewise that
 $K(p-a) + L(p-b) + M(p-c) + N(p-d) + \&c. = 0$; but
 $Kp + Lp + Mp + Np + \&c. = 0$, therefore
 $Ka + Lb + Mc + Nd + Pe + \&c. = 0$.

Because $q = ab + ac + ad + ae + \&c. + bc + bd + be + \&c. + cd + ce + \&c.$; if $q' =$ the coefficient of Kx^{n-3} , then $a(p-a) + q' = q$, or $q' = q - ap + a^2$.

In like manner if q'', q''' , &c. be put for the coefficients of Lx^{n-3} , Mx^{n-3} , &c. then will $q'' = q - pb + b^2$, $q''' = q - pc + c^2$, &c.; hence
 $K(q - pa + a^2) + L(q - pb + b^2) + M(q - pc + c^2) + \&c. = 0$.

But $Kq + Lq + Mq + \&c. = 0$, therefore
 $K(pa - a^2) + L(pb - b^2) + M(pc - c^2) + \&c. = 0$; and it has just been proved that

$K \times pa + L \times pb + M \times pc + \&c. = 0$, therefore
 $K \times a^2 + L \times b^2 + M \times c^2 + \&c. = 0$.

Put r', r'', r''' , &c. for the respective coefficients of Kx^{n-4} , Lx^{n-4} , Mx^{n-4} , &c. Then because $r = abc + abd + abe$, &c. + $bcd + bce$, &c. + cde , &c. + &c. $aq' + r' = r$, or $r' = r - aq' = r - aq + pa^2 - a^3$; in like manner $r'' = r - qb + pb^2 - b^3$, $q''' = r - qc + pc^2 - c^3$, &c. and $Kr' + Lr'' + Mr''' + \&c. = 0$, therefore $K(r - qa + pa^2 - a^3) + L(r - qb + pb^2 - b^3) + M(r - qc + pc^2 - c^3) + \&c. = 0$: but from the above it will appear that $K(r - qa + pa^2) + L(r - qb + pb^2) + M(r - qc + pc^2) + \&c. = 0$, therefore $Ka^3 + Lb^3 + Mc^3 + \&c. = 0$. In the very same

manner we may proceed to $Ka^{n-2} + Lb^{n-2} + Mc^{n-2} + \&c. = 0$; for there are $n - 1$ columns which vanish, but it is evident the last does not. Hence

we have $\frac{x^m \dot{x}}{x^n - px^{n-1} + qx^{n-2} - \&c.} =$

$$\begin{aligned} & K \times \left(a^{n-1} x^{m-n} \dot{x} + a^n x^{m-n+1} \dot{x} \dots \dots + a^{m-1} \dot{x} + \frac{a^m \dot{x}}{x-a} \right) \\ & + L \times \left(b^{n-1} x^{m-n} \dot{x} + b^n x^{m-n+1} \dot{x} \dots \dots + b^{m-1} \dot{x} + \frac{b^m \dot{x}}{x-b} \right) \\ & + M \times \left(c^{n-1} x^{m-n} \dot{x} + c^n x^{m-n+1} \dot{x} \dots \dots + c^{m-1} \dot{x} + \frac{c^m \dot{x}}{x-c} \right) \\ & + \&c. \qquad \qquad \qquad + \&c. \end{aligned}$$

If m be less than n , then $\frac{x^m \dot{x}}{x^n - px^{n-1} + qx^{n-2} - \&c.} =$

$$Ka^m \times \frac{\dot{x}}{x-a} + Lb^m \times \frac{\dot{x}}{x-b} + Mc^m \times \frac{\dot{x}}{x-c} + \&c. \text{ all}$$

the preceding terms vanishing; the fluent in each case is therefore, easily found.

Ex. I. Required the fluent of $\frac{x^5 \dot{x}}{x^3 - px^2 + qx - r}$?

Here $n = 3, m = 5$; therefore, by substitution,

$$\begin{aligned} & + K \times \left(a^2 x^2 \dot{x} + a^3 x \dot{x} + a^4 \dot{x} + a^5 \times \frac{\dot{x}}{x-a} \right) \\ \frac{x^5 \dot{x}}{x^3 - px^2 + qx - r} & = + L \times \left(b^2 x^2 \dot{x} + b^3 x \dot{x} + b^4 \dot{x} + b^5 \times \frac{\dot{x}}{x-b} \right) \\ & + M \times \left(c^2 x^2 \dot{x} + c^3 x \dot{x} + c^4 \dot{x} + c^5 \times \frac{\dot{x}}{x-c} \right) \\ & \qquad \qquad \qquad + K \end{aligned}$$

$$+K \times \left(\frac{a^2 x^3}{3} + \frac{a^3 x^2}{2} + a^4 x + a^5 \times \text{h.l. } \overline{x-a} \right)$$

$$\&f \frac{x^5 \dot{x}}{x^5 - p x^4 + q x^3 - r x^2 + s x - t} = +L \times \left(\frac{b^2 x^3}{3} + \frac{b^3 x^2}{2} + b^4 x + b^5 \times \text{h.l. } \overline{x-b} \right) \\ +M \times \left(\frac{c^2 x^3}{3} + \frac{c^3 x^2}{2} + c^4 x + c^5 \times \text{h.l. } \overline{x-c} \right),$$

Ex. II. Required the fluent of $\frac{x^5 \dot{x}}{x^5 - p x^4 + q x^3 - r x^2 + s x - t}$?

In this example $n = 5$, $m = 3$, therefore the fluent of

$$\frac{x^5 \dot{x}}{x^5 - p x^4 + q x^3 - r x^2 + s x - t} = K a^5 \times \text{h.l. } (x-a) + L b^5 \times \text{h.l. } (x-b) \\ + M c^5 \times \text{h.l. } (x-c) + N d^5 \times \text{h.l. } (x-d) + P e^5 \times \text{h.l. } (x-e).$$

The manner of determining the values of $K, L, M, \&c.$ is pointed out in every book of fluxions.

If n , be a large number, it is obvious that this method of finding the fluent can scarcely ever be made use of.

ARTICLE XXV.

Three Propositions from Lawson.

(To be answered in Number X.)

PROP. XXXIV.

LET AB meet a circle in C and D, and A be without and B within the same, and let the rectangle CAD be equal to the square of AB, together with the rectangle CBD, and through A any line be drawn meeting the circle in E and F, and BE, BF, be drawn meeting the circle again in G and H; then the points A, G, H, are in a right line.

PROP. XXXV.

From the extremes of AB, let two lines AC, BD be drawn to touch a circle in C and D, and in AB let a point E be taken on the same side of A with B, such that the rectangle BAE may be equal to the square of AC, and also in AB another point F on the same side of B with E such that the rectangle EBF may be equal to the square of BD, and through A any line be drawn meeting the circle in G and H, and BG, BH be drawn meeting the circle again in K and L; then the points L, K, F, are in a right line.

PROP. XXXVI.

If from A, the vertex of a triangle ABC, be drawn AD to any point D in the base, and DE be drawn parallel to AC, and DF to AB. I say the sum of the rectangles BAF, CAF will be equal to the square of AD, together with the rectangle BDC.

ARTICLE XXVI.

An old PROBLEM with a NEW SOLUTION.

By Mr. JOHN SURTEES, SUNDERLAND.

PROBLEM.

GIVEN the day of the month, or sun's declination, to find the latitude of the place where the duration of twilight is the least possible.

SOLUTION. Fig. 303, Plate 18.

Let HH represent the horizon, Z the zenith, N the nadir, O'CO the parallel of declination, AOA the circle of twilight, ZBN an azimuth circle passing through

through O' the place or point of the horizon where the sun sets, and O the place where twilight ends.

Then since the portion O'BO of the azimuth circle, is the shortest that can be comprehended between the horizon and circle of twilight, it is evident that O'CO will be the shortest portion of the parallel of declination.

Put t = tangent of BO' = half the depression,
 s = sine of the declination = cosine of SO' , and
 radius = 1.

In the right angled spherical triangle SBO' , as

$$\cosine\ BO' = \frac{1}{\sqrt{1+t^2}} : radius = 1 :: \cosine\ of\ SO'$$

= s : cosine of BS = $s \sqrt{1+t^2}$, and in the right angled spherical triangle SBN , as radius = 1 : cosine BN =

$$\frac{t}{\sqrt{1+t^2}} :: \cosine\ of\ BS = s \sqrt{1+t^2} : \cosine\ of\ SN$$

= st = the sine of the latitude, that is, as radius is to tangent of half the depression, so is the sine of the declination to the sine of the latitude; which is different from what *Emerson* and others have asserted,

N. B. If the latitude is north, the declination must be south, and *vice versa*.

Mr. *Emerson* has erred in his solution of this problem.---The angle pvN (see his figure, page 492, *Miscellanies*) is greater than 90° , when twilight is

shortest, so of consequence its cosine = $\frac{sx-l}{cy} = \frac{l}{y}$,

and $sx = l \times \sqrt{1+c}$, which agree exactly with the above.

ARTICLE XXVII.

Answers to the Mathematical Questions proposed in

ARTICLE X. No. VI.

I. QUESTION 110, answered by Mr. Olinthus Gregory, Book-seller and Teacher of the Mathematics, Cambridge.

IN order to give an accurate solution to this question, it will be necessary to consider the piece of fir, in each of the two instances, as immersed in, and sustained by two media, of different specific gravities; in the first case *air* and *water*, and in the next, *air* and *oil of turpentine*. The following theorem from Mr. MARTIN (which perhaps is not so well known as it ought to be) will then enable us to determine the requisite particulars: "The part of the solid in the heavier fluid, is to the whole solid, as the difference between the specific gravity of the solid and lighter fluid, to the difference between the specific gravity of the two fluids."—Now, the specific gravity of water, oil of turpentine and fir, being as expressed, in the question, and that of air being 1.25, we have this proportion, (considering the contents of prisms on equal bases, as proportional to their altitudes) as 1000—1.25 (difference between the specific gravity of water and air) : 550—1.25, (difference between the specific gravity of fir and air) :: 6 Inches (altitude of the part immersed in water) : 10.92027335 inches, the altitude or side of the cubic piece of fir. Again by means of the same theorem, as 800—1.25 (difference between the specific gravity of oil of turpentine and air) : 550—1.25 :: 10.92027335 inches : 7.50234742 inches, (altitude of the part immersed in oil of turpentine). Had this question been solved according to the common rule, which is only strictly true *in vacuo*, it would have given $10\frac{1}{4}$ inches for the side of the cube, and $7\frac{1}{2}$ inches for the depth of the part immersed in oil of turpentine.

N. B. In questions of this nature, another theorem of Mr. MARTIN's may often be had recourse to with advantage: for the benefit of the young student who is unacquainted with the rule it is here added: "as the part of the solid within the heavier fluid, is to the part contained within the lighter: so is the difference of the specific gravity of the solid and lighter fluid, to the difference between the specific gravity of the solid and heavier." The investigation of these two rules is given by Mr. MARTIN, and would have been repeated here, had it not been considered as a pleasing exercise for the Tyro to trace out the rationale of them himself.

The

The same by Mr. John Byerly, Ripon, Yorkshire.

By VINCE's Hydrostatics, Prop. XVI. we have,

As the specific gravity of fir 550 :
Is to the specific gravity of water 1000 ::
So is the part immersed in water :
To the whole cube ::
So is the depth of the immersed part, 6 inches :
To the side of the cube, $10\frac{1}{11}$ inches.

Again, by the corollary to the same proposition, we have this theorem, viz. "If the same body float upon two different fluids; the parts immersed will be inversely as the specific gravities of the fluids." Hence, as $800 : 1000 :: 6 : 7\frac{1}{2}$, or $800 : 550 :: 10\frac{1}{11} : 7\frac{1}{2}$ inches, the depth immersed in oil of turpentine.

N. B. The part above the fluid is here supposed to be in vacuo. But when a body floats upon a fluid, the part above the fluid, being in the air, this rule will not be accurately true; near enough so, however, for all practical purposes. The effect of air, if necessary, may be computed by the following theorem, viz. "If a lighter fluid rest upon a heavier, and their specific gravities be as $a : b$, and a body, whose specific gravity is c , rest with one part P in the upper fluid, and the other part Q in the lower, then $P : Q :: b - c : c - a$, or $Q : P + Q :: c - a : b - a$. Vide VINCE's Hydrostatics, Prop. XXI.

And thus nearly was the answer given by Mr. Johnston.

The same answered by the Rev. L. Evans, Royal Academy, Woolwich.

Let x = the side of the cube of fir sought, then $550x^3$ = its weight, and $6000x^3$ = the weight of water displaced by the fir floating in it. Therefore $550x^3 = 6000x^3$. Hence $x = 10\frac{1}{11}$ inches, the side of the cube. Then as $800 : 550 :: 10\frac{1}{11} : 7\frac{1}{2}$ inches, the depth immersed in oil of turpentine.

Messrs. Bosworth (the Proposer), Cantabrigus, Harris, Lowry, Marrat, Peacock, Simpson, and Swale answered it exactly like Mr. Evans,

II. QUESTION 117, answered by Cantabrigus.

In this question are given the diameters of a vessel in form of a conic frustum, 18 and 14 inches respectively, and the altitude 28 inches, to find which base must be downwards that the liquor contained therein may be longest in running out of a circular orifice an inch in diameter; and how much longer time the liquor would be exhausting with one end downwards than with the other. Here we

may

may with advantage, have recourse to a very ingenious *Dissertation on the Exhausting of Vessels*, in Article I. of *Hutton's Miscellanea Mathematica*; Corollary 1st to Prob. 2nd of which contains all that is necessary to our purpose.

Let a be put for 28, the altitude of the vessel, $n =$ the area of the orifice, $m = 386$ inches, distance described by falling bodies in the first second, $b = 254.469005$ inches, area of the greater base and $t = 153.93704$, area of the less base. Then, by the theorem

in the corollary above mentioned, $\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{8b+4\sqrt{bt}+3t}{15} =$

time of exhaustion, the greater base being downwards; and

$\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{3b+4\sqrt{bt}+8t}{15} =$ time of exhaustion, the less base being

downwards. The former of these expressions exceeds the latter by

$\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{b-t}{3}$, the value of which is 22.9831 seconds. Therefore

Mr. Gregory's neighbour had better place his cask with the greater end downwards: for then if the spike-peg should be left out, and the cork in the bottom should give way, the cask would be 22.9831 seconds longer in discharging, than it would through an equal aperture in the less base, if it were downwards.

The same answered by Mr. W. Peacock, Land-surveyor, Birmingham.

It is evident that the liquor will run out slower when the greater end is downwards than when the less end is downwards; for, the surface of the liquor will descend with a greater velocity, and consequently the altitude of the issuing fluid will be less, and of course the velocity will also be less. This premised, put $b = 18$, $t = 14$, $n = .7854 =$ also the area of the orifice, $m = 386$ inches, and $a = 28$. Then by *Dr. Hutton's Miscellanea Mathematica*, Prob. 2nd, Cor. 1, the time of the liquors running out

at the greater end will be $\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{8b^2 + 4bt + 3t^2}{15} \times n,$

and at the less end it will be $\frac{2\sqrt{a}}{n\sqrt{m}} \times \frac{8t^2 + 4bt + 3b^2}{15} \times n,$

the difference of which is $\frac{2\sqrt{a}}{\sqrt{m}} \times \frac{b^2 - t^2}{3} = 22.98$ seconds, the time required.

Ingenious answers to this question were also given by Messrs. Bosworth, Evans, Gregory (the Proposer), Johnston, Lowry, Marrat and Swale.

III. QUEST-

III. QUESTION 112, answered by Mr. Gregory, Cambridge.

In fig. 279, pl. 18, let HEDC represent the wall, EC the altitude of which is given 9 feet, on which stands the cylindrical vessel FACD, whose height AC is to be ascertained. Most writers on Hydrostatics and Hydraulics inform us, that if water issue from an aperture B in a vessel the altitude of which is AC, the spouting fluid will form a parabolic curve, the ordinate whereof CI (measured on an horizontal line from the bottom of the vessel) will be double the sine BK of the semicircle described on AC as a diameter: hence it will be easy to ascertain the height of the vessel. Thus, by the nature of the parabola (*Theo. 1, Parab. Hutton's Courfe*), as BE : BC :: EG² : CI²; whence $CI = \sqrt{EG^2 \times BC \div BE} = \sqrt{392 \div 11}$, BC, BE, and EG, being, by the question, equal to 2, 11, and 14 respectively. Now, from what has been observed above, $BK = \frac{1}{2} CI = \frac{1}{2} \sqrt{392 \div 11}$: therefore, by the well known property of the circle, as BC (=2) : BK (= $\frac{1}{2} \sqrt{392 \div 11}$) :: BK : BA = $\frac{1}{2} \times \frac{392}{11} \div 2 = 4\frac{5}{11}$; and consequently $2 + 4\frac{5}{11} = 6\frac{5}{11}$ feet, the altitude AC of the vessel, as required.

The same answered by Mr. John Surtees, Sunderland.

If a heavy body fall from a height = $\frac{1}{2}$ the parameter of a parabola, and the motion, acquired by falling, be changed into an horizontal motion, inasmuch, that the body may begin to move anew downwards, the projected body will describe that parabola.—Therefore, let $x = AB$ the height of the water above the hole B, then by hydrostatics and the nature of the parabola, $4x \times BE = EG^2$, or $4x \times 11 = 14^2$, hence $x = 49 \div 11$, and $2 + x = 6\frac{5}{11}$ feet, the answer.

Ingenious solutions were also given by Messrs. Bosworth, Byerly, Evans, Harris, Johnston, Lowry, Marrat, Peacock, and Swale.

IV. QUESTION 113, answered by the Rev. Mr. L. Evans.

Let $x =$ the length of the timber. Then $x : 3\frac{1}{2} :: 12:42 \div x =$ half the difference between the girts at the less end and at the section required; therefore $\frac{84}{x}$ is = that difference, and therefore

$\frac{84}{x} + 2 = \frac{84+2x}{x}$ = the girt at the section. Hence $\left(\frac{84+2x}{x} + 2\right)$

$\div 2 = \frac{42 + 2x}{x}$ = the mean girt of the piece having the less

end,

end, and $\left(\frac{84 + 2x}{x} + 9\right) \div 2 =$ the mean girt of the piece having the greater end. Consequently

$$\frac{1}{4} \times \frac{42 + 2x}{x} \times 12 + \frac{1}{4} \times \frac{42 + 5\frac{1}{2}x}{x} \times x - 12 = \frac{1}{4} \times 5\frac{1}{2} \times x + 3.$$

Whence $x = 17.81$ feet, the answer.

The same by Mr. John Harris, Land Surveyor, and Teacher of the Mathematics, Caermarthen.

Put $a =$ lesser girt, $= 2$, $b =$ greater girt, $= 9$, $c = 12$ AG (fig. 279, pl. 18) the distance, of the section required, from the lesser end, $e = 3$, $x =$ AH, the length of the piece of timber, and

$n = 3.1416$. Then $\frac{a+b}{2} =$ the girt in the middle of the whole

piece, and the square of $\frac{1}{4}$ th of it $= \frac{(a+b)^2}{64}$, which multiplied by

x (the length) gives $\frac{(a+b)^2}{64} \times x$, for the content of the whole

piece. . . . Now, $\frac{a}{n} =$ the diameter AC, and $\frac{b}{n} =$ the diameter

BD, and by similar triangles n (AH) : $\frac{b-a}{2n}$ (BH) :: c (AG) : $\frac{b-a}{2nx} \times$

$c =$ EG; therefore the diameter EF $(= 2EG + AC) = \frac{b-a}{nx} \times c + \frac{a}{n}$

consequently the girt at the section EF $= \frac{b-a}{x} \times c + a$, and the

girt at the middle of the lesser piece AEFC $= \frac{b-a}{2x} + a$; the square

of $\frac{1}{4}$ th of this is $= \frac{(b-a)^2}{64x^2} \times c^2 + \frac{b-a}{16x} \times ac + \frac{a^2}{16}$ which multi-

plied by c (AG), gives $\frac{(b-a)^2}{64x^2} \times c^3 + \frac{b-a}{16x} \times ac^2 + \frac{a^2c}{16} =$ the

content of the frustum AEFC.

Again,

Again, $\frac{b-a}{2x} \times c + \frac{a+b}{2}$ is the girt at the middle of the greater piece EBD \bar{F} , and the square of $\frac{1}{4}$ th of this is $= \frac{\overline{b-a}^2}{64x^2} \times c^2 + \frac{b^2-a^2}{32x} \times c + \frac{\overline{b+a}^2}{64}$ which multiplied by $\overline{x-c}$ (GH), gives $\frac{\overline{b-a}^2}{64x} \times c^2 + \frac{b^2-a^2}{32} \times c + \frac{\overline{b+a}^2}{64} \times x - \frac{\overline{b-a}^2}{64x^2} \times c^3 - \frac{b^2-a^2}{32x} \times c^2 - \frac{\overline{b+a}^2}{64} \times c$, for the content of the frustum EBD \bar{F} .

Now, the frustum AF $= \frac{\overline{b-a}^2}{64x^2} \times c^3 + \frac{b-a}{16x} \times ac^2 + \frac{a^2c}{16}$, and ED $= \frac{\overline{b-a}^2}{64x} \times c^2 + \frac{b^2-a^2}{32} \times c + \frac{\overline{a+b}^2}{64} \times x - \frac{\overline{b-a}^2}{64x^2} \times c^3 - \frac{b^2-a^2}{32x} \times c^2 - \frac{\overline{a+b}^2}{64} \times c$; their sum $= \frac{\overline{b-a}^2}{64x^2} \times c^3 - \frac{\overline{b-a}^2}{64x^2} \times c^3 + \frac{b-a}{16x} \times ac^2 + \frac{\overline{b-a}^2}{64x} \times c^2 - \frac{b^2-a^2}{32x} \times c^2 + \frac{a^2c}{16} + \frac{b^2-a^2}{32} \times c - \frac{\overline{a+b}^2}{64} \times c + \frac{\overline{a+b}^2}{64} \times x$, which, per question must be $= \frac{\overline{a+b}^2}{64} \times x$ (the content of the whole) $+ c$; therefore, by taking away what is common, &c. we have

$$\frac{b-a}{16x} \times ac^2 + \frac{\overline{b-a}^2}{64x} \times c^2 - \frac{b^2-a^2}{32x} \times c^2 + \frac{a^2c}{16} + \frac{b^2-a^2}{32} \times c - \frac{\overline{a+b}^2}{64} \times c = c.$$

$$\text{Now, } \frac{b-a}{16x} \times ac^2 = (4ba - 4a^2) \times \frac{c^2}{64x},$$

$$\frac{\overline{b-a}^2}{64x} \times c^2 = (b^2 - 2ab + a^2) \times \frac{c^2}{64x},$$

$$\text{and } -\frac{b^2-a^2}{32x} \times c^2 = -(2b^2 + 2a^2) \times \frac{c^2}{64x};$$

and

and their sum $= -(b^2 - 2ba + a^2) \times \frac{c^2}{64x} = -\frac{(b-a)^2}{64x} \times c^2.$

Again, $\frac{b^2 - a^2}{32} \times c = (2b^2 - 2a^2) \times \frac{c}{64},$

$$\frac{a^2}{16} \times c = 4a^2 \times \frac{c}{64},$$

and $-\frac{(a+b)^2}{64} \times c = -(a^2 + 2ab + b^2) \times \frac{c}{64};$

and their sum $= (a^2 - 2ab + b^2) \times \frac{c}{64} = \frac{(b-a)^2}{64} \times c.$

Therefore $\frac{(b-a)^2}{64} \times c - \frac{(b-a)^2}{64x} \times c^2 = c$; hence $x = \frac{(\overline{b-a} \times c)^2}{\overline{b-a}^2 \times c - 64c} =$

$$\frac{49 \times 144}{49 \times 12 - 8 \times 64} = \frac{196}{11} = 17.8181 \text{ feet, the answer.}$$

The same answered by Mr. W. Marrat, Boston.

Put $d = \frac{1}{2}$ the difference of the diameters of the piece of timber, $l =$ the left radius, and $x = LI$, the length of the part remaining after 12 feet have been cut off. Then, by *lim. Δs*, $d : x + 12 :: l : l(x + 12) \div d = Nn$, the length required to complete the cone, also, as $l(x + 12) \div d : 2 :: l(x + 12) \div d + 12 : (2l(x + 12) + 24d) \div l(x + 12)$, the circumference at EF; hence,

$$\left(\frac{2l(x + 12) + 24d}{l(x + 12)} + 2 \right) \div 2 = \text{the mean girt of the part EC,}$$

$\frac{1}{2}$ th of which squared, and multiplied by the length $= 12$, we have

$$\left(\frac{l(x + 12) + 6d}{2l(x + 12)} \right)^2 \times 12 = \text{the cont. Again, } \left(\frac{2l(x + 12) + 24d}{l(x + 12)} + 9 \right)$$

$$\div 2 = \frac{21l(x + 12) + 24d}{2l(x + 12)} = \text{the mean girt of the part BF;}$$

consequently

consequently its content is $= \left(\frac{(11 \div 4)l(x+12)+6d}{2l(x+12)} \right)^2 \times x$:

but the content of the whole piece is $(11 \div 8)^2 \times (x+12)$.

Therefore $\left(\frac{l(x+12)+6d}{2l(x+12)} \right)^2 \times 12 + \left(\frac{(11 \div 4)l(x+12)+6d}{2l(x+12)} \right)^2 \times$

$x = (11 \div 8)^2 \times (x+12) + 3$. Hence $x = 17.818$ nearly, answer.

Solutions to this question, agreeing in result with the above, were given by Messrs. Bosworth, Gregory, Johnston, Lowry, Peacock and Swale,

V. QUESTION 114, answered by the Rev. L. Evans, the Proposer.

The business in questions of this kind is, to find what each spirit is worth when reduced to proof, which is the standard of its worth. On this principle the answer is easily obtained. Thus,

If $100 : 110 :: 18 : 19.8s.$
If $100 : 118 :: 10.5 : 12.39s.$ } per gallon proof.

Difference $7.41s.$ B's cheaper than A's per gallon.

The same by Mr. Byerly, Ripon.

As $118 : 18 :: 1 : \frac{9}{59}$, and $1 - \frac{9}{59} = \frac{50}{59}$ the quantity of spirit in one gallon,

therefore $\frac{50}{59} : 10\frac{1}{2} :: 1 : 12.39s.$ the price per gallon of B's.

Again, $110 : 10 :: 1 : \frac{1}{11}$, and $1 - \frac{1}{11} = \frac{10}{11}$, the quantity of spirit in one gallon,

therefore $\frac{10}{11} : 19.8 :: 1 : 19.8s.$ the price per gallon of A's.

Hence $19.8 - 12.39 = 7.41s.$ B's cheaper than A's per gallon.

The same by Mr. William Marrat, Teacher of the Mathematics, Boston, Lincolnshire.

As 110 : 10 :: 4 : $\frac{4}{11}$ gallons of water, and $4 - \frac{4}{11} = \frac{40}{11}$ gallons of spirit; then reciprocally $4 : 18 :: \frac{40}{11} : 19.8s.$ the price per gallon of the brandy at 18s.

Again, As 118 : 18 :: 4 : $\frac{36}{59}$ gallons of water, and $4 - \frac{36}{59} = \frac{300}{59}$ gallons of spirit, and reciprocally, $4 : 10\frac{1}{2} :: \frac{300}{59} : 12.39s.$

the price per gallon of the brandy at 10s. 6d. Hence $19.8 - 12.39 = 7.41s.$ the answer required.

Answers to this question were also given by Messrs. Bosworth, Gregory, Johnson, Lowry, Swale, and Surtees.

VI. QUESTION 115, answered by Mr. J. Hartley, Auditor's Office, Somerset-Place, London.

Fig. 281, pl. 18. Let D, E, F, be the three points the men start from, whose distances are given, and likewise per question, $DG : EG :: FG : EF$, and the angle $DGE = 2EGF$. At the Lemma to Prob. 21, of *Simpson's Algebra*, it is demonstrated, "If a given right line DE be divided in any given ratio, at C, and the right line CF be taken to DC in the ratio of EC to DC — CE; and from F, as a centre at the distance FC, the circle CGO be described, and two right lines DG, EG be drawn from D and E, to meet any where in the periphery thereof; these lines will be to one another (every where) in the given ratio of DC to CE." Whence $DG : EG :: DC : CE$, from which two proportions it is easily proved that, $DG = 2EG$, and DC, CE having the same ratio to each other as DG to GE, it will appear, by Euclid VI. 3, that GC bisects the angle DGE, and that $CGE = DGC = EGF$; consequently $CG = FG$, and GE perpendicular to DF; whence, by Euclid VI. 13, $\sqrt{CE \times EO} = GE = 173.24$ yards, and $DG = 346.48$ yards; also by construction $CF = 200$ yards.

The same answered by Mr. John Lowry, Birmingham.

Let D, E, F be the points the men start from; then there is
given

given $DE = 300$, $EF = 100$; also $DG : EG :: FG : EF$ (not DF as printed), and the angle $DGE = 2EGF$. Make $CE = EF$, and with the distance FC and centres F and C describe arcs to intersect in G , the point required. Join DG , CG and FG , join also GE , which will be perpendicular to DF . Now the $\triangle FCG$ being equilateral, $DC = CG = CF$, and $\angle CDG + \angle DGC = \angle FCG = \angle CGE + \angle EGF$, hence it is evident that the $\triangle s$ DGF , EGF , are similar; therefore, the angles DGC , CGE , EGF , are equal, and $DGE = 2EGF$, and also $DG : GE :: GF : EF$.

Calculation. $GE = \sqrt{GF^2 - EF^2} = \sqrt{200^2 - 100^2} = \sqrt{30000} = 173.20508076$, $DG = \sqrt{DE^2 + EG^2} = \sqrt{300^2 + 30000} = \sqrt{120000} = 346.41016151$, and $FG = 200$ yards, the distances required.

This Question was also answered by Messrs. Bosworth, Gregory, and Sanderford, the Proposer.

VII. QUESTION 116, answered by Mr. Lowry.

Let CA , CB , fig. 282, pl. 18, be the asymptotes to the given hyperbola VTX ; describe the opposite and conjugate hyperbolas QFK , uGn , and WEY . Suppose AB drawn as required; and draw the tangents BEH , AGI , IFH , also join FT , GE ; the figure $ABHI$ is a parallelogram, and $GE = 2CE$, is equal and parallel to AB (*Dr. Hutton on the Hyperbola, Pr. 24. Cor. 4.*), therefore CE is given. Hence this.

Construction. From the centre C , to the conjugate hyperbola WEY , apply $CB =$ half the given line, and draw the tangent EB , meeting CB in B . From B , to AC , apply AB of the given length, and it will touch the curve at T .

N. B. The given line must not be less than the conjugate axis.

If PZ be a circle whose centre is C , and radius any line CP , less than the semitransverse axis of the hyperbola, a right line may easily be drawn to touch both the circle and hyperbola. For the triangle ACB is of a constant magnitude, being one fourth part of the parallelogram $ABHI$, that is, one fourth part of the rectangle under the axes, and CP is given, and perpendicular to AB , therefore, AB is given, and the construction will be the same as above.

Other ingenious constructions nearly similar to the above were given by Messrs. Swale and R. Wallace, the proposer.

VIII. QUESTION 117, answered by Mr. Johnston, Birmingham.

Analysis. Let ACB (fig. 283, pl. 18.) be the \triangle to be constructed,

ed, CAD the given angle at the base, and CD the given bisecting line. Since the $\triangle ACB$ is to be a maximum, the $\triangle ACD$, being the half thereof, must be a maximum; consequently, by Theo. 6, Simpson on the Max. et Min. $AD = AC$. Hence the construction is evident.

Similar to this was the answer by Mr. Gregory.

The Composition by Mr. R. Elliott, Liverpool.

On the given bisecting line, as a base, construct the isosceles $\triangle DCA$, having the angle at $A =$ the given one; make $BD = AD$, and join BC , so shall ACB be the \triangle required.

For CD bisects the base AB , the angle at A is $=$ the given one, and the isosceles $\triangle DCA$ is a maximum, therefore its double, or the $\triangle ACB$ is also a max.

Mr. Louis Hill, constructs the Problem generally, viz. When the \triangle is of a given magnitude, and the given line divides the base into segments having to each other a given ratio. His construction is as follows:

On the given bisecting line CD (fig. 284, pl. 18.) let a segment of a circle be described, capable of containing the given angle. Make the rectangular parallelogram $DEFC$ equal to twice the given area, and divide DE in the given ratio at Q ; draw QB parallel to DC , and let it meet the circle in B . Join DB , BC , and continue BD so that AD may be to BD in the given ratio; join also AC , and ACB will be the triangle required.

Demon. Draw AP perpendicular to CD , and produce BQ to meet CF in G .

By sim. $\triangle s$, $AP : QD :: AD : DB :: QE : QD$, therf. $AP = QE$.

Whence, it is evident, that, the $\triangle DCB$ is equal to half the parallelogram $DCGQ$, and the $\triangle CDA$ equal to half the parallelogram $QGFE$, therefore the whole $\triangle ACB$ is equal to half the parallelogram $DEFC$, that is, equal to the given magnitude.

The \triangle will evidently be a maximum when QB touches the circle at B , the $\triangle CBD$ being then isosceles

Constructions were also given by Messrs. Harris, Lowry, Peacock, Swale, and Thornoby.

IX. QUESTION 118; answered by Mr. Lowry.

Put $a = 60$ feet the length of the plane, $b = 45$ feet, the distance of the lowest part of the plane from the surface of the water,

$s = \text{fine } 40^\circ$, to radius 1, and $g = 16 \frac{2}{12}$ feet. Then, by Mecha-

tics,

nics, the force which accelerates the centre of gravity of the ball, while it rolls down the plane, is $\frac{5}{7}g$, and its velocity at the end of the plane is $= \sqrt{(20 \div 7) \times gsa}$, or the same as would be acquired by descending perpendicularly through the space $(5 \div 7) \times sa$, therefore the velocity with which the ball enters the fluid is $= a$.

$\sqrt{g \times (5 \div 7) \times sa + b} = 68.26$ feet per second. Now put $N = 925$ the density of the ball (being oak, and not copper, as printed), $n = 1.000$ that of the fluid, $d = 1$ the diameter of the globe, $m = 3\pi \div 8Nd = \frac{15}{37}$, $V = 68.26$, the first velocity, $x =$

the space descended in any time t , and v the velocity at the end of that time.

Then, by Prob. 21, page 355, Vol. II. *Dr. Hutton's Course of Mathematics*, $mx = \text{Hyp. Log. } \frac{V}{v}$, or $x = \frac{1}{m} \times \text{Hyp. Log. } \frac{V}{v} =$

$\frac{37}{15} \times 4.22332 = 10.417522$ feet the depth of the Reservoir.

N. B. The resistance which the ball meets with from the air in descending from the top of the plane to the surface of the water is not taken into consideration in the above solution; the effect being so very inconsiderable. The velocity, however, in that circumstance is easily determined from Prob. 22, page 350, Vol. II. of *Dr. Hutton's Course*; for, if s be put $= (5 \div 7)sa + b$, $c = 2.718281828$, $w = 2g \div m$, then the velocity is $=$

$$\sqrt{w - wc^{-2ms}}.$$

Solutions similar to this, and agreeing with it in result, were received from Messrs. Bosworth, Byerly, Gregory, and Marrat.

The same by Mr. John Surtees, Sunderland.

Let $s =$ sine of the given angle to radius 1, $a = 60$ feet $=$ length of the inclined plane, $h = 45$ feet, and $g = 16 \frac{1}{12}$: Then

$a\sqrt{gsa} =$ the velocity of the ball at quitting the plane, that is, the velocity acquired by falling down the plane; and by the nature of projectiles, $a\sqrt{g \times (sa + h)} =$ its velocity just going into the water. Hence, by *Emerson's Fluxions*, page 384, the depth of the Reservoir,

$$\text{Reservoir, } x = \frac{4dg \times 2.302585}{3p} \times \text{Log.} \frac{q-p \times 16dg - 3pb^2}{q-p \times 16dg - 3pv^2}$$

where d = diameter of the globe of Oak = 1 foot, q = 925 its density or specific gravity, p = 1000 the density of the fluid, g = $16\frac{1}{12}$ feet the space through which a heavy body descends by gra-

vity in one second, t = time of describing x , v = the velocity at the end of the time t , and b = the velocity with which the ball enters the fluid = $2\sqrt{g \times (sa + h)}$. Therefore x may readily be determined.

X. QUESTION 119, answered by Mr. Thornoby.

Fig. 285, pl. 18. Let ABC be the required Δ ; put $AB = a$, the given side, d = the different of the segments of the base, and $x = AD$ the greater segment; then $x - d$ = the less segment, and $BD^2 = a^2 - x^2$; but $AB^2 - BC^2 = AD^2 - DC^2 = 2xd - d^2$, therefore $BC^2 = a^2 - 2xd + d^2$; and by the question $AB \times DC$ is to $BC \times BD$ in a given ratio; suppose that of m to n . Hence $a^2 (x - d)^2 : (a^2 - 2xd + d^2) \cdot (a^2 - x^2) :: m^2 : n^2$, and by multiplying means and extremes, we have $2dm^2 x^3 - ((a^2 + d^2)m^2 - n^2 a^2) \times x^2 - (2da^2 (m^2 + n^2)) x = n^2 a^2 d^2 - m^2 a^4 - m^2 a^2 d^2$, a cubic equation, which shews that the Problem cannot be constructed geometrically, that is, by right lines and circles; but recourse must be had to the conic sections.

Messrs. Gregory, Lowry, and Johnston sent observations of the same nature as the above.

XI. QUESTION 120, answered by Mr. Lowry, Birmingham.

Let ADB (fig. 286, pl. 18.) represent the glass in its inclined position, PB the surface of the liquor; then there is given $DE = 4$, and $\angle BPC = 63^\circ$ (not 36° as printed in the question), also, by the question, the solidity of the segment DPB must be equal to half the paraboloid ADB. Draw PC perpendicular to AB; then (by Cor. 4. pa. 379, *Dr. Hutton's Mensuration, new Edition*) $(BC^2 \div AB^2) \frac{1}{2} DE \times .785398$, is equal to the content of the segment DPB, and $AB^2 \times \frac{1}{2} DE \times .785398$, is equal to the content of the paraboloid ADB; therefore $2BC^2 \div AB^2 = AB^2$; hence $AB = \sqrt[4]{2 \times BC}$, and $AC = (\sqrt[4]{2} - 1) \times BC$. Put s , and c , for the sine and cosine of 63° , then will $PC = \frac{c}{s} BC$, and, by Prop. II. Cor. Dr. Hutton

on the parabola, $DE : PC :: AE^2 :: BC \times AC$, or $DE : \frac{c}{2}$

$BC :: \frac{1}{2} \sqrt{2} \times BC^2 : (\sqrt{2} - 1) \times BC^2 :: \sqrt{\frac{2}{16}} : \sqrt{2} - 1$;

hence, $BC = DE \times \frac{5}{c} \times (\sqrt{2} - 1) \times \sqrt{8} = 4.293249$, $AB = 5.117$, and the content of the glass 41.126 solid inches.

W. W. R.

Ingenious solutions, to this question, were also received from Messrs. Gregory, Johnston, Marrat, and Thornoby.

XII. QUESTION 121, answered by Mr. Lowry.

Put $a = 4$ lb. the weight of the ball, $f = 1000$ lb. the elastic force of the spring, $b = 3$ inches $= \frac{1}{4}$ of a foot, the distance the spring is bent, when the velocity of the striking body is destroyed, $g = 16\frac{1}{12}$ feet, $x =$ the space the spring is bent at any time t , $v =$ the velocity at that time, and $z =$ the velocity of the ball when it first strikes the spring. Now, by experiments made upon springs, it appears that the action of the spring, is as the compressing force, or in a constant ratio to the distance x ; therefore $b : f :: x : fx \div b$, $=$ the force of the spring when it is bent through the distance x : but, by the doctrine of motion, $-v = 2gfx \div bau$, or $-vv = 2gfx \div ba$, the fluent of which corrected by making $v = z$, is $z^2 - v^2 = 2gfx \div ba$. Now, when $x = b$, v is $= 0$, therefore $z = \sqrt{2gfb \div a} = 44.83767$ the velocity of the ball when it first strikes the spring, and the distance the ball must descend to acquire this velocity is $z^2 \div 4g = 31\frac{1}{4}$ feet, the height required.

Again, the time of descending the distance $z^2 \div 4g$ is $z \div 2g = 1.393917$ seconds; and to find the time of the springs bending 3 inches we have $i = x \div v$: but it is evident, from above, that $v = \sqrt{z^2 - (2gfx \div ab)}$, and $f = az^2 \div 2gb$, therefore, $v = \sqrt{z^2 - z^2 x^2 \div b^2}$; and $i x \div v = b x \div z \sqrt{b^2 - x^2}$, and the correct fluent, when $x = b$, is $(b \div z) \times$ quadrantal arc of a circle, whose radius is unity, $= .008758$; hence, the whole time of descending is 1.402675 seconds.

The

The same answered by Mr. Surtees, Sunderland.

Let $w=4$ lb. $c=1000$ lb. $a=16\frac{1}{2}$ feet, $b=\frac{1}{4}$ ($=3$ inches), and d = the height required. Then, by Prob. 21, page 405, Emerson's Fluxions, $4ad=2a(cb \div w$; hence $d=b \times (c-2w) \div 2w=31$ feet = the height required, and $\sqrt{b(c-2w) \div 2aw}=1.39$ seconds = the time of descent.

According to one or other of these methods were the solutions given by Messrs. Byerly, Gregory, Johnston, Marrat, Peacock, and Thornoby.

XIII. QUESTION 122, answered by Mr. Lowry.

Fig. 287, pl. 18. Let N be the North Pole; B, Berwick; P, Penzance; D, Dover; and B, Birmingham. Draw the great circles PN, PB, B'N, B'B, BD, BN, and DN. Now in the spherical $\triangle B'NB$, there are given, the $\angle B'NB$ = the diff. of longitude between Berwick and Birmingham = $0^{\circ} 5'$; NB = the colat. of Berwick = $34^{\circ} 12'$; and NB' = the colat. of Birmingham = $37^{\circ} 30'$; to find BB'. In the spherical $\triangle PNB'$, there are given, the $\angle PNB'$ = the diff. of longitudes between Penzance and Birmingham = $4^{\circ} 10'$; B'N = $37^{\circ} 30'$; and PN = the colat. of Penzance = $39^{\circ} 52'$; to find PB'. And in the spherical $\triangle B'ND$, there are given the $\angle B'ND$ = the diff. of longitude between Birmingham and Dover = $3^{\circ} 3'$; B'N = $37^{\circ} 30'$; and DN = the colat. of Dover = $38^{\circ} 53'$; to find BD.

Let O be the centre of the earth, draw OB', OP, OB, and OD, and produce OB' to a , so that Ba may be = 300 feet, the height of the telegraph at Birmingham. Also draw the tangents acd , age , qfb , to touch the circles at e , g , f , and to meet the radius of the earth, produced, at d , c , b ; then are Pd, Bc, Db, the heights of the telegraphs at Penzance, Berwick, and Dover respectively. Join Oe, which will be perpendicular to ad ; then in the right angled plane \triangle , there are given Oe = the earth's radius, and Oa = Oe + 300, to find the $\angle eOa$; now the arch B'P, or the measure of the $\angle dOa$, being given, the $\angle dOe$ is easily found. Hence dO, and consequently Pd, the height of the telegraph at Penzance is readily had. And in the very same way are the heights of the telegraphs at Berwick and Dover found.

A method of solution similar to the above was pointed out by several of our Correspondents: but very few of them have attempted to work out their solutions; and of those who have, no two agree in their conclusions. Not being able to rely upon any received, we have merely given the method pointed out by one of our correspondents as above.

XIV. QUESTION 123, answered by Mr. Thornoby.

Let AC (fig. 288, pl. 18), represent the candle, GFE, NMS, sections of the given globes, and let lines be drawn as in the figure. Suppose A the point required, then it is evident that the shadows of the globes, upon the table, will be ellipses, whose transverse axes are BD, and LR and conjugates the shadows of chords drawn perpendicular to GF, MN, through the points H, V.

Since it will be extremely tedious to give a direct solution to this question, we had better have recourse to the method of approximation by trial and error. Thus, assume AE of any convenient length, then ES being given = 10 feet, AC = 13, OE = 6, and IS = 4 inches, AI becomes known, and so do PC, CO, CK, and AK, by sim. Δ s and Eu. I. 47, also, by trigonometry, the angles AKC, BKC, OCG, OCF are known; hence, in the Δ s DCK, KCB, CK and all the angles are given to find DK and KB; then DK + KB, the transverse axis, will be given. Again, by sim. Δ s, GH, and CH, are easily found; hence, as CH : sGH :: CK : to the conjugate axe. Therefore, the area of the ellipsis becomes known. And in the very same way may the area of the other ellipsis be found. Now, if the two areas happen to be equal, the question is solved; if not, assume some other number for AE, and proceed as before, till you find the area of the ellipsis; note the results, and errors, and find the value of AE by the common method of trial and error.

A similar method was pointed out by other Correspondents.

XV. QUESTION 124, answered by Mr. Marrat, Boston.

In fig. 289, pl. 18, EFI being a section of the equilateral cone, there are given HB = $1\frac{1}{2}$, CH = 4, and the content of the frustum EKL^F is to be a maximum. Now this is equal to the difference between the cones EIF, and KIL, and, by the property of the equilateral cone, their solidities have a constant ratio, to the cubes of their sides, or to the cubes of their altitudes; therefore it is obvious, that, when the frustum EKL^F is a maximum, or when the quantity of water displaced is the greatest possible, the difference of the cubes of IH, and IG, will be a max. This premised, put $x = IG$, then GF is $= x \div \sqrt{3}$, and by the property of the parabola, as $HG^2 = 2\frac{1}{2} : HC = 4 :: GF^2 = \frac{1}{3} x^2 : (16 \div 27) x^2 = CG$, hence $HI = x + (16 \div 27) x^2 - 4$; therefore $x^3 - (x + (16 \div 27) x^2 - 4)^3$ must be a maximum.

Which put into fluxions and reduced gives $x^3 + \frac{405}{96} x^4 - \frac{53945}{6912} x^5$:

$$-\frac{8478}{256}x - \frac{888 \times 10683}{24576}x + \frac{48 \times 10683}{24576}. \quad \text{Hence } x = \frac{5}{4}\sqrt{3}$$

nearly; and the weight of the cone is 23.589 ounces, avoirdupois.

The same answered by Mr. Harris, Caermarthen.

Let ACB represent the glass, EIF the cone, and put CH = a , the parameter of the parabola = p , and GF = $\frac{1}{2}$, the side of the cone = x . Then from the property of the equilateral Δ , GI = $x\sqrt{3}$, and from the nature of the parabola, CG = $x^2 \div p$; therefore GH = $a - x^2 \div p$, and HI = $x\sqrt{3} + x^2 \div p - a$. And by sim. Δ s, $x\sqrt{3}$ (FG) : x (GF) :: $x\sqrt{3} + x^2 \div p - a$ (IH) : HL = $(px\sqrt{3} + x^2 - ap) \div p\sqrt{3}$. Now, by mensuration, the solidity of the frustum, KLEF = $(EF^2 + KL^2 + EF \times KL) \times GH \times .2618 = (8 \times (px\sqrt{3} + x^2 - ap) \div p\sqrt{3})^2 + 4x^2 + (8 \times (px\sqrt{3} + x^2 - ap) \times 2x \div p\sqrt{3})) \times (a - x^2 \div p) \times .2618$, a max. per quest. which fluxed and reduced, gives $x^2 + \frac{5}{2}p\sqrt{3}x^2 - (2ap - 6p^2)x^2 - 3ap^2\sqrt{3}x^2 + (a^2p^2 - 3ap^2)x + \frac{3}{2}a^2p^2\sqrt{3} = 0$. Hence $x = 1.287$, &c.

This question was also answered by Messrs. Bosworth, Gregory, Johnston, Lowry, and Peacock.

XVL. QUESTION 125, answered by Mr. Gregory, Cambridge.

When the globe sunk so deep as to conceal $\frac{2}{5}$ of its surface, $\frac{2}{5}$ of its altitude was immersed, because the surfaces of spheric segments are as their heights: hence $\frac{2}{5}$ of 15, or 6 inches of the globe's axis were sunk in the water. Hence it is easy to find the solidity of the globe = 1767.15 inches, and that of the segment immersed = 622.0368 inches. Now, it is evident from the principles of Hydrostatics that the weight of the whole globe is equal to the weight of the water displaced by the segment immersed: that is, as 1728 : 1000 :: 622.0368 : 359.975 ounces, weight of the globe. The content of the upper segment of the globe is 1767.15 - 622.0368 = 1145.1132 inches. The weight of its bulk of water is 662.68125 ounces; which, it is obvious, must be the weight of the ball of copper in water, necessary completely to immerse the given globe. Then by the nature of specific gravity, we have, as 8 (difference between the specific gravities of copper and water) : 9 (specific gravity of copper) :: 662.68125 : 745.5164 ounces, weight of the globe of copper in air. Therefore, as 9000 : 1728 :: 745.5164 : 143.13196 inches, content of the copper globe: whence $\sqrt[3]{143.13196 \div .5236} = 6.490134$ inches, its diameter, as required.

The

The same by Mr. William Peacock, Birmingham.

Because the surfaces of segments of a sphere are to each other, or to the surface of the whole sphere, as their heights, $\frac{2}{3}$ ths, or 6 inches, of the globe's axis will be immersed in the fluid, and by hydrostatics, the weight of the globe is equal to the weight of water displaced by it, that is, equal to 22.448lb. Again, the weight of the copper in water, must evidently be equal to the weight of water displaced by the upper segment of the globe, that is, equal to 41.417578lb. Then, by the law of specific gravities of bodies, as the difference between the specific gravities of copper and water, is to the specific gravities of copper, so is the weight of the copper in water, to its weight in air = 745.5164 ounces. Hence the diameter of the copper ball will be 6.49 inches.

Ingenious answers to this question were also received from Messrs. Bosworth, Byerly, Francis, (the Proposer), Johnston, and Thornoby.

XVII. QUESTION. 146, answered by Mr. Harris, Caermarthen.

Analysis. Suppose ACB (fig. 290, pl. 18.) the Δ to be constructed, O the centre of the inscribed circle GHI. Draw EF perpendicular to AB, bisecting it in L, and meeting a circle described through the points A, O, B, in E and F; draw also EP perpendicular to AO, and ON to EF, and join EO.

Since, OH (=OG = OI), and the $\angle OCH$, ($=\frac{1}{2}\angle C$), are given, CO is given: but CO + AO + BO, is given; therefore AO + BO, and consequently OP ($=\frac{1}{2} \times AO + BO$), is given.

Again, the $\angle AOB$ ($= \text{right } \angle + \frac{1}{2}\angle C$) is given, therefore the $\angle POE$ ($=\frac{1}{2}\angle AOB$) is given, and therefore EP is given.

Hence the rect. EL \times EN = (by Cor. 4. Pr. VI. Art. LII. Vol. I.) EP², and LN = OG is a given line; therefore EN is given, hence the diameter EF is given, and the construction is evident.

The same answered by Mr. Lowry, Birmingham.

Since the radius of the inscribed circle, and the vertical angle are given, the distance between the centre of that circle and the vertex will be given; therefore, the sum of the distances from the centre of the inscribed to the angular points at the base of the triangle is given. Hence this

Construction.

Construction. Fig. 291, Pl. 18. Make the $\angle AOB =$ half the vertical \angle plus a right \angle , and take $OQ =$ half the sum of the distances from the angular points at the base to the centre of the inscribed circle. Drop the perpendicular QD and let it meet OD , drawn to bisect the $\angle AOB$ in D , and on OD demit the perpendicular QP , and, from D , apply DF , to QP , so that $DF \times (DF + \text{given radius})$ may be $=$ to the square of QD . Through F , draw AFB perpendicular to DF , and let it meet AO , OB in A and B , and draw AC to make the $\angle OAC = \angle OAB$, and BC to make the $\angle OBC = \angle OBA$; so shall ACB be the triangle required.

Demon. Continue DF to meet OE , drawn parallel to AB , in E , and AO to meet BC in I . Upon AB demit the perpendicular OR . Because the \angle s BAC , ABC , are bisected by the lines AO , BO , O is the centre and OR the radius of a circle inscribed in the $\triangle ABC$. But, by the Corollary to Theo. 19, B. IV. Simpson's Geometry, and similar \triangle s $QD^2 = OD \cdot DP = FD \cdot (DF + EF) = FD \cdot (DF + \text{given radius})$ by the construction, therefore $OR = EF =$ the given radius.

Again $AO + OB - QOQ =$ the given sum of the distances from the angular points at the base to the centre of the inscribed circle.

And the angle $AOB = OIB + OBI = ACB + IAC + OBI$. But $\frac{1}{2} ACB + IAC + OBI =$ a right angle. Therefore the $\angle ACB$ is $=$ the given one. Q. E. D.

The same by Mr. William Wallace, Perth.

Analysis. Suppose ABC (fig. 292, pl. 18.) the triangle required, and that ABC is the given vertical angle. Let D be the centre of the inscribed circle; join DA , DB , DC , and draw DE , DF , perpendicular to AB , BC , the sides about the given angle. Because, by hypothesis, DE , or DF , the radius of the inscribed circle is given, as also the angle ABC , it is evident that DB , which bisects the angle ABC , is given: now $BD + AD + DC$ is given, by hypothesis, therefore $AD + DC$ is given. Draw HK perpendicular to DB , meeting AB in L , and CD , produced in K . The angle KLA , or $B LH$ is evidently half the sum of the angles BAC , BCA , and the angle KDA is equal to the sum of the angles DAC , DCA , that is, to half the sum of the same angles BAC , BCA ; therefore the angles KLA , KDA , are equal, and the points K , L , D , A are in the circumference of a circle, hence the angle DKH is equal to the angle DAE ; now the angles at H and E are right angles, and $DH = DE$, therefore $DK = DA$, and CK is equal

to $AD + DC$, that is, to a given line: this problem is, therefore, reduced to the following well known problem, which has been enunciated, and solved, in various ways by most writers in geometry. Two straight lines HL , BF , being given by position; it is required to apply between them a straight line KC of a given length, which may pass through D , a given point equally distant from these lines.

Construction. With the given radius of the inscribed circle, let a circle be described whose centre is D ; draw the radii DE , DF , so that the angle EDF may be equal to the supplement of the given vertical angle: draw BA , BC touching the circle at the points E , F ; thus the angle ABC will evidently be equal to the given vertical angle. Let np be the given sum of the distances of the angles of the triangle from the centre of the inscribed circle: join BD meeting the circle in H ; draw HL perpendicular to BD ; take $pq = BD$, and between the lines BF , HL apply KC equal to np , so as to pass through D , the centre of the circle.* Draw CA touching the circle at G and meeting BE at A , and ABC shall be the triangle required, as is evident from the Analysis.

Remark. The preceding Analysis contains the demonstration of the following Theorem, which may, in certain cases, be applied with advantage to the construction of problems relating to triangles.

THEOREM.

If from D , the centre of a circle inscribed in a triangle, straight lines DA , DB , DC be drawn to the angles, and LO be drawn, touching the circle at H , the point where BD , one of these lines cuts the circle, and if AD , CD , the other two lines, be produced to meet the tangent in K and M : the lines CK , AM shall be equal to one another, and to the sum of AD , DC .

Corollary. AL is equal to KO , and CO is equal to LM .

Messrs. Gregory, Johnston, Peacock, and Thornoby sent ingenious constructions to this question.

XVIII. QUESTION 127, answered by Nobody.

XIX. QUESTION 128, answered by Cantabrigus.

In order to solve this curious question, let us first endeavour to
VOL. II. R ascertain

* The manner of doing this is shewn in Prob. 21, *Sim'son's Geometry*, 3rd Edit. also in Prop. 32, Lib. I. of *D'Omerique's Geom. Anal.* and else where.

ascertain how high a building may be raised at the Equator before the stones are carried off its top, by reason of the centrifugal force overcoming the centripetal. And here it seems necessary to find the velocity of a body revolving in a circle at any given distance from the Earth's centre, by means of its own gravity. Writers on central forces determine this in the following manner: Put $g = 16 \frac{1}{2}$ feet, the space through which a body descends by the force of gravity in one second at the surface of the earth, and $r =$ the radius of the earth; then the velocity with which a body must be projected at the earth's surface to revolve round it, like a satellite, may be found by this theorem $\sqrt{2gr} =$ (in this case, supposing 25000 miles the earth's circumference) 26000 feet per second: [See Dr. Hutton's *Mathematical and Philosophical Dictionary*, Article CENTRAL Forces; and Thorp's *Commentary on Newton's Principia*, Prop. 4. Lib. 1. Pa. 85.]. Then, since the force of gravity varies in the inverse duplicate ratio of the distance, we have as $\sqrt{R} : \sqrt{r} ::$

26000 : $\sqrt{\frac{r}{R}}$; the velocity of a body revolving round the earth

at any distance R . And $5078 \sqrt{\frac{R^3}{r^3}} = T$, the time of revolution in the same: where 5078' is the periodic time of a body revolving at the surface. From the nature of our present enquiry it may be inferred, that when 23^h 56' (the periodic time of the earth's rotation on its axis) is substituted for T , in the last theorem,

if R denote the distance of the top of the building from the centre of the earth, that theorem will enable us to discover the height of the building so that its top shall be carried round in the same time by the earth's diurnal rotation, as it would by the combination of the central forces. Thus, if $5078 \sqrt{\frac{R^3}{r^3}} = 86160$, then $R =$

$$\sqrt[3]{r^3 \times \frac{86160^3}{5078^3}} = 6.603017. \text{ Hence } R = r = 5.60301 \text{ times}$$

the earth's radius, the height of the building at the equator, so that the stones will just remain on its top: for if they were laid any higher they would fall off, and be projected round the earth in the same manner as the moon.

SCHOLIUM.

Considering the earth as a sphere, this problem may be readily made general. Thus, in Fig. 293. Pl. 18, let P and p denote the poles

poles of the earth, Pp the axis, EQ part of the equator, and $EB = R$ in the foregoing theorems. It is required to find Ab the greatest height to which a building can be raised in any latitude AQ . It is evident Cb is the radius of the circle described in the earth's revolution by b the top of the building; which must in all cases be equal to $EB = R$. Hence, as sine of CEb ($=$ cosine of latitude) : Cb ($= R$) :: radius ($=$ sine of bCE) : Eb , from which deduct the radius of the earth, and the remainder will be $Ab =$ the height of the building.

Corollary. As the latitude AQ increases the height Ab will increase; because the cosine of the latitude which is the first term in the proportion, decreases: and at the pole P , the height cannot be determined; because the first term in the proportion will have vanished.

The same answered by Mr. Surtees, Sunderland.

Let $r =$ radius of the earth in feet $= 3980 \times 1760 \times 3$, $g = 16\frac{1}{2}$, $t = 3.1416$, $s =$ cosine of the latitude to radius 1, and $T = 23^h 56^m = 86160$ seconds $=$ the time of one revolution, also $R =$ height from the centre to the top of the building. Then, by Dr. Hutton's *Mathematical and Philosophical Dictionary*, Vol. I. Pa. 261, when the centrifugal force is an exact balance for gravity c

$\sqrt{\frac{27}{g}} \times \sqrt{R^2 \div r^2} = T$, at the equator, and for any particular

latitude $c \sqrt{\frac{27sR^2}{g r^2}} = T$, hence $R = \sqrt[3]{\frac{T^2 r^2 g}{27s^2}}$, and $R - r =$

the height required.

For London, $s = .6229$ nearly, and $R - r = 141503016$ feet, $=$, nearly, 26800 miles $=$ the height to which a building might be raised at London before the stones would leave the top by reason of the centrifugal force being an over balance for gravity.

XX. QUESTION 129, answered by Mr. William Wallace, of Perth Academy.

Figures 294, 295, 296, 297, 298, Plate XVIII. Let A, B, C be the three given points, and DEF the given circle, it is required to inscribe in it a triangle, the sides of which may pass through the given points.

R 2

CASE

CASE I. Fig. 294. First suppose the points to lie in a straight line*.

Analysis. Suppose that the triangle DEF is inscribed in the circle, and that its sides pass through the given points A, B, C, as is required. Draw EG parallel to AB, meeting the circle again in G; and join FG, meeting AB in H: the angle AHF is equal to the angle EGF, or to ADF, therefore, the points A, D, H, F are in the circumference of a circle, and the rectangle AB·BH is equal to the rectangle DB·BF, now the point B, and the rectangle DB·BF are both given, therefore the rectangle AB·BH is given, and since the point A is given, the point H is also given. Draw EK touching the circle at E, and meeting AB in K: the angle KEF is equal to EGF, or to KHF, therefore, the points K, E, H, F are in the circumference of a circle, and the rectangle KC·CH is equal to EC·CF, now the point C, and the rectangle EC·CF are given, the point H has also been shewn to be given, therefore, the point K is given, and since EK is a tangent to the given circle, the point E is given; hence the following construction:

Construction. That the problem may be possible it is evident that either one of the given points, or all the three must lie without the given circle. Draw a straight line through B, one of the given points, so as to meet the circle in L and M: in AB take the point H, so that the rectangle AB·BH may be equal to LB·BM, and so that the points A, H, may be on the same side, or on contrary sides, of the point B, according as the points L, M are on the same side, or on contrary sides of the point B. Let a straight line passing through C meet the circle in L and M, and take K in AB, so that the rectangle HC·CK may be equal to LC·CM, and so that the points H, K may be on the same side, or on contrary sides of C, according as the points L, M are on the same side, or on contrary sides of C. Draw KE touching the circle at E, join CE meeting the circle again at F, join BF meeting the circle again at D, and join AD; the straight line AD shall pass through E, and EDF shall be the triangle required.

Demonstration. From the points E, F inflect EG, FG to the circumference of the circle. Because the rectangle KC·CH is equal

* This case is indeed excepted in the question, but as it admits of a construction considerably more simple than the most general case of the problem, it has been presumed that a particular consideration of the excepted case would be an improvement to the answer.

† This will always be possible, for it may be shewn that the point K is without the circle; whatever be the situation of the points, provided that either one only, or all the three be without the circle.

equal to $LC \cdot CM$, that is, to $EC \cdot CF$, the points K, E, H, F are in the circumference of a circle, therefore, the angle AHF , or KHF is equal to KEF , that is, to EGF : again, because the rectangle $AB \cdot BH$ is equal to $LB \cdot BM$, or to $DB \cdot BE$, the points A, D, H, F are in the circumference of a circle, therefore, the angle ADF is equal to AHF , that is, as has been shewn, to EGF : now the angles at D and G , being equal, must stand upon equal circumferences, therefore, AD will pass through the point E , as was to be demonstrated.

CASE II. Fig. 295. Next let A and B , two of the given points, be in LM the diameter of the given circle, and let C the remaining point be without the diameter.

Analysis. Suppose that DEF is the triangle required. Draw EG parallel to the diameter, meeting the circle again in G , and join FG , meeting the diameter in H : the angle AHF is equal to EGF , or to ADF , therefore, the points A, D, H, F are in the circumference of a circle, and the rectangle $AB \cdot BH$ is equal to $TB \cdot BD$, that is, to a given space, now the points A, B are given, therefore, the point H is also given. Draw FO through O the centre of the circle, and join NE meeting AB in K : the angle KHF is equal to EGF , or to KNF , therefore, the points K, N, H, F , are in the circumference of a circle, therefore, the rectangle $KO \cdot OH$ is equal to $NO \cdot OF$, or to the square of the radius, which is given; now the points O, H are given, therefore the point K is given; and because NF is the diameter, the angle KEC is a right angle, and is, therefore, given: Join KC , and the point E will be in the circumference of a given circle, viz: that which has KC for its diameter, but it is also in the circumference of the given circle DEF , therefore, the point E is given.

Construction. Take the point H in AB , so that the rectangle $AB \cdot BH$ may be equal to $LB \cdot BM$, and so that the points A, H may be on the same side, or on contrary sides of B , according as the points L, M lie on the same side, or on contrary sides of B : Also take the point K so that the rectangle $KO \cdot OH$ may be equal to the square of LO , the radius of the given circle, and so that the points K, H may be on different sides of the centre O ; join KC , and upon KC as a diameter describe a circle, which, when the problem is possible, will cut the given circle in E ; join CE meeting the circle again in F , and join BF meeting the circle again in D ; join AD ; the straight line AD shall pass through E , and DEF shall be the triangle required.

Demonstration. Join KE which produce to N , and join FN ; because the angle KEF is a right angle, the line FN is the diameter, and since by construction the rectangle $KO \cdot OH$ is equal to the square of LO , or to $NO \cdot OF$, the points K, N, H, F are in the

circumference of a circle, and the angle KNF , or ENF is equal to KHF , and because the rectangle $AB \cdot BH$ is equal to $LB \cdot BM$, or to $DB \cdot BF$, the points A, D, H, F are in the circumference of a circle, and the angle ADF is equal to AHF , which has been shewn to be equal to ENF : because the angles ADF, ENF are equal, they stand upon equal circumferences, therefore, AD must pass through E , and the sides of the triangle EDF which is inscribed in the circle pass through the given points as was required.

CASE III. Fig. 296. Let A and B , two of the given points, be in LM , the diameter of the circle, and so situated, that $LA : AM = LB : BM$.

In this case it is well known that if AD, BD be inflected to any point in the circumference of the circle, and produced to meet it again in E and F ; the line which joins the points E and F shall be perpendicular to LM the diameter of the circle. Therefore, if through the point C (which must be within the circle) a perpendicular be drawn to the diameter, meeting the circumference in E and F , and FB be joined, meeting the circle in D , the straight line which joins the points A, D shall pass through E , and EDF shall be the triangle required.

Before proceeding to Case IV. it will be convenient to premise the following Lemma:

LEMMA. Figure 299, Plate XVIII.

Let HTR be a straight line drawn from any point H in LM the diameter of a circle, meeting the circle in T and R , draw RV perpendicular to the diameter, meeting the circle in V , join HV meeting the circle in Q , and join RQ , meeting the diameter in K ; MH shall be to HL as MK to KL . Also, let two points H, K be taken in LM , the diameter of a circle, so, that $MH : HL = MK : KL$: draw any straight line through K , meeting the circle in R and Q , join HR and HQ meeting the circle in T and V : the arch LR shall be equal to LV , and LT shall be equal to LQ .

As the truth of both parts of this lemma is pretty generally known, we shall not add a demonstration, but proceed to the fourth Case.

CASE IV. Fig. 297. Next let the three given points be any how situated.

Analysis. Suppose that DEF is the triangle required, the sides of which are to pass through the given points A, B, C . Join AB , and draw EG parallel to AB , meeting the circle in G ; join FG , meeting AB in H : it will appear as in the first two cases, that the rectangle $AB \cdot BH$ is equal to $DB \cdot BF$, that is, to a given space, therefore,

therefore the point H is given. Let O be the centre of the given circle, join HO meeting the circle in L and M, the line HO is given by position, and the lines HL, HM are given. Draw FN perpendicular to LM, meeting the circle in N, and join NH, meeting the circle in P, join FP meeting OH in K; by the lemma, $MH : HL = MK : KL$, therefore, the point K is given. Join BK, meeting the circle in R and Q; the line BK, and the points Q, R are evidently given by position; join AR meeting the circle in S, and the point S will also be given. Join HQ meeting the circle in V and HR meeting the circle in T, and join SV. It has been shewn that the rectangle $AB \cdot BH$ is equal to $DB \cdot BF$, that is, to $RB \cdot BQ$; therefore, the points A, R, Q, H are in the circumference of a circle; hence the angle BHQ is equal to QRA, that is, to QVS, therefore, SV is parallel to AB, or EG, and the arch VG is equal to the arch SE. Now (by the Lemma) the arch LP is equal to LG, and LR to LV, therefore, the arch PR is equal to the arch GV; but GV was shewn to be equal to ES, therefore, PR is equal to ES, and the arch PE is equal to the arch RS. Now the points R, S, and, therefore, the arch RS were shewn to be given, therefore, the arch PE, and the angle PFE, or KFC are given in magnitude; and since the points C, K, are given, the point F is in the circumference of a given circle, which will pass through the points C, K; but it is also in the circumference of the given circle DEF, therefore, the point F is given, and the line EF is given by position, therefore, also the straight lines AED, BFD, are given by position.

Construction. Let A and B be any two of the given points: join AB and draw any straight line BR, meeting the circle in Q and R; take H in AB, so that the rectangle $AB \cdot BH$ may be equal to $RB \cdot BQ$, and so that the points A, H may be on the same side, or on contrary sides of B, according as the points Q, R are on the same side or contrary sides of B. Let O be the centre of the given circle, join HO, meeting the circle in L and M, and take K in LM, so that $MH : HL = MK : KL$: join BK, meeting the circle in Q and R, and join AR, meeting the circle in S: join CK, and upon CK describe a segment of a circle that may contain an angle equal to any angle, at the circumference of the given circle, which stands upon the arch RS, let this circle meet the given circle in F, join CF, meeting the given circle at E, join BF, meeting the circle at D: the straight line which joins the points A, D, shall pass through E, and the sides of the triangle DEF, which is inscribed in the given circle, shall pass through the given points A, B, C, as is required.

Demonstration. Join FK meeting the circle in P, draw HPN, HTR, HQV, HGF, and join EG and SV. Because the rectangle

angle

angle $AB \cdot BH$ is equal to $RB \cdot BQ$, the points A, R, Q, H are in the circumference of a circle, therefore, the angle BHQ is equal to QRA , that is, to QVS , therefore, SV is parallel to AB ; and because $MH : HL = MK : KL$, the arch LR is equal to the arch LV ; also LP is equal to LG (by the Lemma), therefore, PR is equal to GV ; but, by construction, the arch PE is equal to RS , therefore, PR is also equal to ES ; wherefore, ES is equal to GV , and the straight line EG is parallel to SV , or to AB . Because, by construction, the rectangle $AB \cdot BH$ is equal to $RB \cdot BQ$, or to $DB \cdot BF$, the points A, H, F, D are in the circumference of a circle, therefore, the angle FDA is equal to BHF , that is, (because GE is parallel to AB), to HGE , but the angle HGE is evidently equal to any angle at the circumference of the circle and standing upon the circumference FGE , therefore, the angle FDA must stand upon the circumference FGE , therefore, DA passes through E , and the sides of the triangle DEF , which is inscribed in the given circle, pass through the given points A, B, C , as required.

Besides the four preceding cases, in which it is easy to see that the points might be so situated as to render the problem impossible, there is yet a fifth case, in which the problem, instead of becoming impossible, may become unlimited, or admit of numerable answers. This will be evident from the following Theorem, which, together with its demonstration, will be found in Dr. Stewart's *Propositiones Geometricæ*, Lib. I. Prop. 33, and 34.

THEOREM. Fig. 298.

In LM the diameter of a circle, take a point H , and let HQ be perpendicular to the diameter; let A and B be two points in HQ , on the same side of H , if that point be within the circle, but on contrary sides if it be without the circle, and let the rectangle $AH \cdot HB$ be equal to the rectangle $LH \cdot HM$; from the points A, B inflect straight lines AD, BD to D any point in the circumference, meeting the circle in E and F , join EF , meeting LM in the point K : MH shall be to HL as MK to KL .

Suppose now that A and B , two of the given points in our problem, are situated as required in the hypothesis of the above proposition, it is evident that the point K will be given, and since the remaining point C is supposed to be also given, the straight line ECF will be given by position; hence the construction is obvious.

But if we suppose the points C and K to coincide, then, it is evident that the problem would become indeterminate; for if any straight line whatever be drawn through K , meeting the circle in E and F , and if AE and BF be joined, the lines AE and BF shall meet at D a point in the circumference of the given circle.

It

It seems proper to acknowledge, that the view here given of this curious geometrical problem was suggested by the following Porism, which appears to have been the last of the Third Book of *Euclid's Treatise on Porisms*.

"If from two given points A, B, any two straight lines AD, BD be inflected to the circumference of a circle given by position, meeting it again in E and F, the straight line which joins E and F shall either contain a given angle with a straight line passing through a given point, or shall be parallel to a straight line given by position, or shall pass through a given point."

A complete investigation of this Porism may be seen in Dr. Simson's valuable work, entitled *De Porismatibus*.

The same answered by Mr. Lowry, Birmingham.

A general solution to this problem may be derived from the converse of *Lawson's 7th Proposition*, the demonstration of which is given in the *Third Number of the Repository*. Let D, E, F (fig. 305, pl. 19.) be the three given points, ABC the circle which is given in magnitude and position, S its centre, and ABC the required triangle. Through any two of the given points, as D, E, draw KDEI to intersect the circle in K and L, and meet BQ, drawn so as to make the $\angle BQE = ACB$, in Q; then DE, KE, and EL, are given lines; and by reason of the equal angles, the triangles DCE, EBQ are similar; therefore $DE : CE :: EB : EQ$, and by the circle $KE : CE :: EB : EL$, therefore, $DE : KE :: EL : EQ$, consequently EQ is given, and therefore, Q is a given point. Through the given points Q, S, draw QCSH to meet the circle in C and H, then HQ, GQ, and HG are given lines, and if I be a point in HQ, such, that $HQ : GQ :: HI : GI$ (as in *Lawson's Prop.*), then I is a given point. Now let IE, IB, be drawn to cut the circle in P, O, and R, and PQ, OD, to cut it in T and N respectively; join OT, NT, AM, and RM (M being the point where BQ intersects the circle); then by the *Proposition above cited*, RM and OT are both perpendicular to HQ, consequently they are parallel. Again, it is shewn above, that $DE : KE :: EL : EQ$, and by the circle $KE \cdot EL = OE \cdot EP$; therefore, $DE : OE :: EP : EQ$, consequently the triangles EDO, EPQ, having the angles at E equal, are equiangular (Euclid. 6. VI.); therefore $\angle TQK = PON = QTN$, and since $\angle AMB = ACB = BQE$, it follows that TN and AM are each parallel to KL, and by reason of the parallels RM, OT, the $\angle ABR = AMR =$ the supplement of the $\angle NOT$. Now I, E and D, being given points, O and N are also given points, and, therefore, the $\angle NOT$ is given, and consequently its supplement, or the $\angle ABR$ is given; hence
the

the problem is reduced to this, *viz.* to draw the lines AFB, RIB, to pass through the given points F, I, meet in the circumference of the given circle, and include a given angle, the method of effecting which is well known.

XXI. Or PRIZE QUESTION 130, answered by Mr. John Lowry, Birmingham.

Figure 300, Plate 18. Let A, B be the given points, bisect AB, in C, and take $CD = CE =$ half the length of the string, the curve will evidently pass through the points D and E. Let F be any other point in the curve, and describe the great circles AF, BF, and on AB demit the perpendicular arch FP. Put $a =$ the sine, and $c =$ the cosine of AB, $b =$ the sine, and $d =$ the cosine of DE, $x =$ the cosine of PB, the ordinate, and $y =$ the cosine of PF, the abscissa of the curve required.

Then the sine of PB $= \sqrt{1 - x^2}$, and by trigonometry

Radius(1):Cosine of PB (x):Cosine of PF (y):Cosine of BF $= xy$;

hence, sine of BF $= \sqrt{1 - x^2 y^2}$; theref. by cor. to Prop. 2. Simp. Trig.

Cosine of AP $=$ Cos. AB \times Cos. PB \pm Sine AB \times Sine PB, that is,

the Cosine of AP is $= cx \pm a\sqrt{1 - x^2}$; and the

Cos. of AF $=$ Cos. (AF + FB) or DE \times Cos. BF \pm Sine DE \times Sine BF, that is,

the Cosine of AF is $= dxy \pm b\sqrt{1 - x^2 y^2}$;

Hence, by Trigonom. Rad. (1):Cos. AP::Cos. FP:Cos. AF, or

$1 : cx \pm a\sqrt{1 - x^2} :: y : dxy \pm b\sqrt{1 - x^2 y^2}$, or by reduction,

$b\sqrt{1 - x^2 y^2} = (d - c) \times xy \pm ay\sqrt{1 - x^2 y^2}$, an equation expressing the nature of the curve.

When the length of the string is equal to half the circumference of a great circle of the sphere, the Locus will be a great circle whose pole is C.

For draw CF, and continue the great circles FA, FB, FC, till they meet at G, which will be at a semi-circle's distance from the point F; then since AC = CB, and AF \pm FB = AG \pm GB = a semicircle, it follows that CF is = CG = $\frac{1}{2}$ quadrant, but CD = CE is also = a quadrant, therefore the locus of the point F is a given circle whose pole is C.

Hence, we are led to the discovery of many beautiful PORISMS, or general Propositions, which are of great use in the construction of Spherical Problems; the following are selected from a variety of others on the same subject.

PORISM I. Fig. 301, Plate 18.

Let there be two great circles CAD, CBD, given by position, a point P may be found, such, that if through it any great circle ACB be drawn to meet the other great circles at A and B; the sum of the arches AC, CB, will always be equal to a certain given arch.

Draw the great circle CPD, so as to bisect the angle ACB, and meet the circles given by position at D, (which will be at a semi-circle's distance from C); bisect the arch CPD in P, the point to be found; that is, if through P any great circle APB be drawn, the sum of the arches AC, CB is equal to a given arch. For the angles ACP, BCP, ADP, BDP are all equal to one another, and the angle APD is \equiv CPB, also the arch PC \equiv PD, therefore the triangles APD, CPB are every way equal, and therefore the arch CB is \equiv AD. In the same way it is shewn that the arch AC is \equiv BD. Therefore $AC + CB \equiv AD + DB \equiv CAD \equiv CBD$ \equiv a semicircle, that is, equal to a given arch.

PORISM II. Fig. 302, Plate 18.

Let there be three great circles given by position intersecting each other at the points A, C, B, another great circle DE, and also a point P; may be found, such, that if through the point P any great circle PQV be drawn, to intersect the other circles in Q and V, the spherical trapezium DAVQ will be equal to the spherical trapezium VBEQ.

Find the point P as in Porism. I, and on the great circle FPI, drawn perpendicular to AB, take PI \equiv PF, and draw DIE perpendicular to PI; then P is the point, and DIE the circle which were to be found.

It is easily shewn that the spherical spaces DAFI, BFIE are equal, and the right angled spherical triangles QPI, FPV are also equal, therefore, it is evident, that the spherical trapezia DAVQ, VBEQ are equal.

PORISM III. Fig. 303, Plate 18.

Let there be two great circles CAD, CBD given by position, a point P may be found, such, that if through it any great circle APB be drawn to intersect the other circles in A and B, the angles CAP will always be equal to the angle DBP.

PORISM

PORISM IV.

Let there be any two great circles given by position, another great circle, and also a point, may be found, such, that if through that point any great circle be drawn to intersect the given ones; and from the points of intersection, and likewise from the point found, great circles be drawn to the circle found; the sum of the three arches so drawn will be equal to a given arch.

The same answered by Mr. Richard Elliott, Liverpool.

Fig. 304, plate 18. Let A and B be the given points, and AEB the string; moreover let DF be the principal axis of the curve, and from any point E, let EC be drawn perpendicular to DF. Put the sine of AD = s , and its cosine = c , the sine of BD = m , and its cosine = a , the cosine of CD = x , and the cosine of CE = y . Then, by Trigonometry, $s\sqrt{1-x^2} + cx = \text{cosine of AC}$, and $m\sqrt{1-x^2} + mx = \text{cosine of BC}$. Hence, since the triangles AEC, BEC, are right-angled at C, we have the cosine AE = $sy\sqrt{1-x^2} + cxy$, and the cosine of BE = $my\sqrt{1-x^2} + axy$. Now, the Cosine of the sum of the arches AE, BE, will be $(sy\sqrt{1-x^2} + cxy) \times (my\sqrt{1-x^2} + axy) - \sqrt{1-(sy\sqrt{1-x^2} + cxy)^2} \times \sqrt{1-(my\sqrt{1-x^2} + axy)^2} = n$, the cosine of the length of the string.

If the arch CD = 0, its cosine, x , will be = unity, and the equation of the curve will become, by proper reduction, $(a^2 + c^2 - 2acn) \times y^2 = 1 - n^2$; hence, it appears that when the arch CD = 0, the arch CE is = to a given quantity, and, therefore, that the curve will be convex at the point D, in all cases except when $a^2 + c^2 - 2acn = 1 - n^2$, or $y = 1$.

If the two points A and B coincide, a will be equal to c , and s equal to m , and the equation in that case becomes $(sy\sqrt{1-x^2} + cxy)^2 - 1 + (sy\sqrt{1-x^2} + cxy)^2 = n$, or $sy\sqrt{1-x^2} + cxy = \sqrt{\frac{n+1}{2}}$, as it ought, being the equation to a circle on a sphere.

The same answered by Mr. Gompertz, London.

Fig. 300, pl. 18. Put s and c = sine and cosine of AB, m and n =

π = sine and cosine of $AF + FB$, x = sine of PB , and y = sine of PF , all to radius 1. By trigonometry, cosine $AP = sx + c\sqrt{1-x^2}$, cosine of $BF = \sqrt{1-y^2} \times \sqrt{1-x^2}$, and cosine of $AF = m\sqrt{1-(1-y^2) \times (1-x^2)} + n\sqrt{1-y^2} \times \sqrt{1-x^2}$.

Also, by trigonometry, as $\cos. AP : \text{rad.} :: \cos. AF : \cos. PF$, that is, as $sx + c\sqrt{1-x^2} : 1 :: m\sqrt{1-(1-y^2) \times (1-x^2)} + n\sqrt{1-y^2} \times \sqrt{1-x^2} : \sqrt{1-y^2}$.

Hence $\frac{sx}{\sqrt{1-x^2}} + c = \frac{m\sqrt{1-(1-y^2) \times (1-x^2)}}{\sqrt{1-y^2} \times \sqrt{1-x^2}} + n$, the equation required.

And thus it was answered by Messrs. Furness, Harris, Surtees, and Thornoby.

The Medal for solving the Mathematical Prize Question is decided in favour of Mr. Lowry, who will please to send for it to Mr. Glendinning's, by whom it will be delivered, free from expence; to any part in London.

ARTICLE XXVIII.

MATHEMATICAL QUESTIONS.

To be answered in Number X.

I. QUESTION 157, by Mr. Olinthus Gregory.

THE young Algebraist is requested to determine the dimensions and area of a right angled triangle, whose base, perpendicular, and hypotenuse are x^{5x} , x^{6x} , and x^{7x} respectively.

II. QUESTION 158, by Mr. John Byerly, Ripon.

It was observed, on the top of one of the high mountains in Italy, that the atmosphere was only half as dense as it was at the bottom. Query the height of the mountain?

III. QUESTION 159, by Mr. W. Burdon, Acafter-Malbis.

Required a theorem for determining the diameter of an air balloon.

foam, to raise a given weight, to a given height above the earth's surface, the weight of each square yard of the substance composing it, and the specific gravity of the air with which it is filled, being given?

IV. QUESTION 160, *by Mr. R. Simpson, Croxdale, Durham.*

The weight of a cylindrical pillar of marble is two tons. Query its dimensions, when it just supports itself from falling, its side making an angle of 60° with the horizon?

V. QUESTION 161, *by Mr. Gregory, Cambridge.*

A surveyor is requested to lay out 5A. 1R. 24P. of land in the form of a right angled triangle, of such dimensions, that the distance from one extremity of the hypóthense to the point where the inscribed circle touches it, shall be 6 chains. Being unable to find the dimensions of the triangle, as required, he begs the favour of some ingenious gentleman to tell him the length of each side thereof?

VI. QUESTION 162, *by Mr. W. Marrat, Boston.*

Being desirous to know what quantity of water was contained in a glass sphere, whose diameter is 1 foot, and thickness of the glass $\frac{1}{200}$ of an inch, I immersed it in a vessel of water, and found it sunk, till the surface of the water in the sphere, was on a level with the surface of the water in the vessel. How much water was in the sphere, the specific gravity of glass and water being 2600 and 1000?

VII. QUESTION 163, *by Mr. Thomas Milner, Lartington.*

Why does Mr. Emerson in the investigation of Prop. X. B. III. Principia, use $AF \times SA : AB \times SF$ instead of $Aa - Ef : Aa - Bb$?

VIII. QUESTION 164, *by Mr. Hewitt, Teacher of the Mathematics, Bunkill-Row, London.*

A tower 70 feet high cast a shadow 81 feet long, west 3° southerly, at 24 min. past 8 in the morning. Required hence the day of the month (in the spring 1798) and the latitude of the place?

IX. QUESTION 165, by Mr. George Brown, Teacher of the Mathematics, Howdon-Pans, North-Shields.

One morning in May. 1798, I observed the sun to rise exactly at 4 o'clock, and at 6 o'clock, the same morning, I found his altitude was $15^{\circ} 35'$. Required the latitude of the place and the day of the month, by a simple equation?

X. QUESTION 166, by Mr. Samuel Thornoby:

If a vessel in the form of a frustum of a cone, whose top diameter is 16, bottom diameter 4, and depth 20 inches, be filled with water to the depth of 10 inches; it is required to determine how high the water will rise if a globe of 12 inches diameter be let fall into the said vessel.

XI. QUESTION 167, by Mr. Thomas Bulmer, Teacher of the Mathematics, Sunderland, Durham.

At what distance from the greater end of a piece of fir in form of a frustum of a square pyramid, must a prop be placed, so as it may just hang in equilibrio, when a weight of 30lb. is hung at the greater end, and one of 10lb. at the less; the side of the greater end being nine inches, that of the less six inches, and the length six feet.

XII. QUESTION 168, by Mr. John Surtees, Sunderland.

Being at sea on June 21, 1798, in a brisk gale, at nine knots or miles (60 to a degree) per hour, at 4h. 30m. A. M. per watch, I observed the true altitude of the sun's centre = $22^{\circ} 28'$, then standing away between south and east, at 9h. 50m. 4f. A. M. per watch, (the watch being supposed to keep true time between the observations,) I found the true altitude of the sun's centre = $33^{\circ} 18'$, and his bearing due south. Required the true latitudes of the two places of observation, their difference of longitude, and the course the ship steered, admitting the earth to be a perfect sphere?

XIII. QUESTION 169, by Mr. John Blackwell, Hungerford.

On May the 12th 1799, in latitude $51^{\circ} 23'$ north, at nine o'clock, I observed an elliptical picture frame, which I suspended in the sun shine, by a thread connected at the extremity of the

conjugate axis, to make several revolutions about the said conjugate axis, the shadow of which formed several variable spaces on the plane of the horizon, but the greatest of which was 172.8 square inches, from whence it is required to find the area of the frame; the transverse to the conjugate axis being as 3 to 2?

XIV. QUESTION 170, by Mr. Peacock, Birmingham.

Required the area of the greatest ellipsis that can be inscribed in a parabola, whose base is 5 and abscissa 4, when the transverse axis is parallel to the base of the parabola; and also the area of the greatest parabola that can be inscribed in an ellipsis whose axes are 5 and 4?

XV. QUESTION 171, by Mr. Gregory, Cambridge.

During how long an interval will it be impossible that there should be any primary Rainbow, on July 1, 1800, in latitude 52° : and during how long an interval, on the same day, in the same place, can there be no secondary bow?

XVI. QUESTION 172, by Mr. Burdon, Acaster Malbis, York.

Given the radii of the inscribed and circumscribed circles, to determine the Δ when the sides are in geometrical progression.

XVII. QUESTION 173, by Mr. Fletcher, Hollinwood, near Manchester.

In a given circle ACBD to inscribe a Trapezium ACBD such, that $AD = DB$; and if from the angular points A, B, C, D; perpendiculars Ae, Bg, Ch, and Di be drawn to the diagonals CD and AB, and the points e, i, g, h, e, be joined, the continual product of the sides of the trapezium *eighe* and those of the trapezium ACBDA; or $AC \times CB \times BD \times DA \times ie \times ig \times gh \times he$, may be a maximum.

XVIII. QUESTION 174, by Mr. Geo. Sanderfon, London.

In Euler's Theory of Ships, translated by Watfen, art. 21, page 104. If there be three tangents in geometrical progression, the first of which is given equal $\frac{b}{a}$; the difference of the other two
arches

arches will be a maximum, when the sine of twice the greater arch is equal to half the sine of thrice the mean or less of the two unknown arches; required the investigation?

XIX. QUESTION 175, by Mr. J. H. Swale, *Chester*.

Given the base, and one of the angles at the base, to construct the plane Δ , when the sum, or difference of the squares of the sides, together with the square of the perpendicular, is equal to a given space.

XX. QUESTION 176, by Mr. Johnston, *Birmingham*.

Given the ratio of the sides, the radius of the circumscribing circle, and the prolongation of the perpendicular, from the vertical angle upon the base, to the periphery of the circumscribing circle, to construct the plane Δ .

XXI. QUESTION 177, by Mr. Lowry, *Birmingham*.

Given the radius of the circumscribing circle, and the difference of the squares of the sides, to construct the plane Δ , when the area is a maximum.

XXII. QUESTION 178, by Mr. Lowry.

Given the base and the two points therein, where the perpendicular and the diameter of the circumscribing circle, drawn from the vertex, intersect it, to construct the plane Δ .

XXIII. QUESTION 179, by Mr. Louis Hill.

Given the difference of the segments of the base made by the perpendicular, the difference of the segments of the base made by the line bisecting the vertical angle, and the difference of the cubes of the sides, to construct the plane Δ .

XXIV. QUESTION 180, by Mr. John Lowry.

Two circles being given in magnitude and position, and also a point in the line joining their centres; it is required to draw a line, through that point, to intersect the circles, so that the intercepted chords may have a given sum or difference.

XXV. QUESTION 181, by Mr. Wallace, *Perth*.

Let AB, AC be two straight lines given by position, let B and C be given points in these lines, and let P be a given point without them.

them. It is required to draw two straight lines PD, PE meeting the given lines in D and E, so that the angle DPE may be of a given magnitude, and so that the rectangle BD \times CE may be equal to a given space?

XXVI. PRIZE QUESTION 182, by Mr. S. Thornoby.

I have a cylindrical vessel, the diameter of the base of which is 2 feet, and its depth 1 foot. Now supposing this vessel were full of water, and my eye placed perpendicularly above the centre, at 6 inches above the surface, the bottom would appear, not *flat*, but concave: if the bottom of the vessel were in the same shape and position as it appears to my eye in that situation, how much less water would it then contain than it does in reality?

ARTICLE XXIX.

HINTS

RELATIVE TO

FRICITION IN MECHANICS.

By Mr. REUBEN BURKOW.

HYPOTHESIS.

IN the following estimation of friction, the weight or force necessary to overcome the resistance, &c. is supposed to be proportional to the pressure.

OF FRICITION IN THE INCLINED PLANE.

Let AB be an inclined plane, (Fig. 305, Pl. 19.) and let PR represent a weight sustained on it by any force Rm, acting in the direction Rm; and draw PD perpendicular to AB, and let Rm meet PD in n: Now as Rn represents the force that would be necessary to sustain the body, exclusive of friction, and Pn represents the pressure against the plane, if nt be drawn perpendicular to PD meeting it in t, then will nm be the force necessary to overcome the friction in that direction, and Pt the real pressure against the plane AB, when the whole force Rm, necessary to overcome both the weight and the friction, acts in the direction Rm; and as the force nm is equivalent to nt and tm, and nt has no other effect than to alter the pressure, therefore tm is the only force which overcomes the resistance of friction; and as this force is as the pressure, therefore tm is proportional to Pt, and hence the locus of all the points m is a right-line.

Again,

Again, suppose the body, instead of being drawn along, to be sustained at rest only upon the plane; this, it is evident, will require a less force than the other, because the friction prevents the body in part from descending. (Fig. 306, Pl. 19.) Let Rm be the force required, and let the same construction be made as before; then because Rn is the force that would be necessary if there was no friction, mn is the effect of the friction itself; but mn is equivalent to the forces mt and tn ; and as Pn would be the pressure exclusive of friction, Pt is the pressure inclusive; and as the force lost is as the friction, and mt is as the force lost, therefore mt is as Pt , for the friction is as the pressure; consequently the locus of all the points m is a right line passing through P , and making the same angle as DPQ in the former case, and only differing by being drawn on the contrary side of PD .

SCHOLIUM.

In what follows, the force requisite to sustain any body is considered under three different distinctions. First, when it is just barely sufficient to overcome the weight and resistance arising from friction, and the body is considered as just beginning to move in the direction of the force applied, and the force in this case is called the *moving force*: secondly, when this force is diminished till the body would begin to move or descend in a contrary direction if the force was diminished farther; this last I call the *suspending force*; and it is plain that whatever force is applied to the body less than the moving, and greater than the suspending force, the body will remain at rest: lastly, it is manifest that there is an intermediate state, in which such a degree of force may be applied, that the friction will have no effect either way; and this force is the same as would keep the body in equilibrio if there was no friction, because the effect or tendency of friction is to keep the body at rest, or prevent it from moving either way; this being premised, there will be little difficulty in the following.

PROBLEM I.

Having given the weight of the body to be sustained, the inclination of the plane, and the ratio of the friction to the pressure; to find the force requisite to sustain the weight in a given direction.

In the foregoing figures, draw PR and PD at right angles to the horizon and plane respectively, PR representing the weight; take PD to DQ as the pressure to the friction, and let DQ be taken upwards or downwards as the requisite force is motive or suspensive; join PQ , and draw the line Rm in the given direction meeting PQ in m ; then Rm is the force required.

Corollary

Corollary 1. If the friction be the π part of the pressure, and W be the weight, s and c the sine and cosine of the plane's elevation, then the moving force parallel to the plane will be $W(s + c : \pi)$, and the suspending force $W(s - c : \pi)$.

Corollary 2. If the direction of the force be parallel to the horizon, and t be the tangent of the plane's elevation, then $W(t\pi + 1) : (\pi - t)$ will be the moving force, and $W(t\pi - 1) : (\pi + t)$ the suspending force, and Wt the force excluding friction.

Example. If the weight be a ton, the friction $\frac{1}{4}$ of the pressure; $AB = 5$, $BC = 3$, and $AC = 4$, then the moving force will be 3235 pounds, the suspending force 747 pounds, and the force excluding friction 1680 pounds; nearly.

PROBLEM II.

Given the weight of the body, the inclination of the plane, and the ratio of the friction to the pressure; to find the direction so that the sustaining force may be a given quantity, or the least possible.

Draw DQ and QP as before, and let PR be to Rm as the weight to the given force; then from the centre R , with a distance equal to Rm , intersect PQ in m ; then Rm is the required direction when the force is given; but to have it the least possible, draw Rm at right angles to PQ , then Rm is the direction required.

Corollary 1. An expression for the sustaining force when the least possible, may be found as follows: In the triangles PDQ , RQm , the angle Q is common, therefore $PQ : PD :: RQ : Rm$; but PD is a fourth proportional to AB , AC , and PR , and DQ is to PD as 1 to π , supposing this the given ratio; also RD is a fourth proportional to AB , BC , and PR , consequently RQ is equal to DQ either added to or subtracted from DR , as it is the first or second case; and because $PQ : PD :: \sqrt{(\pi\pi + 1)} : \pi :: RQ : Rm$, therefore $Rm = PR (\pi \cdot BC = AC) : AB \sqrt{(\pi\pi + 1)}$ or $(\pi s = c) W : (\sqrt{\pi\pi + 1})$ by substituting s and c for the natural sine and cosine of the plane's elevation, and using the negative or affirmative sign as the force required, is the moving or suspending one respectively.

Example. If $AB = 5$, $BC = 3$, and $AC = 4$, and the weight a ton, then the least moving and sustaining forces will be 1825 and 702 pounds respectively.

Corollary 2. Because the triangles PDQ and RQm are similar, and the ratio of PD to DQ constant to each fixed value of π , therefore the angle QRm being equal to DPQ , will also be constant, whether the inclination of the plane be variable or not; and hence the angles of the direction with the plane for the draught to be made with the greatest advantage, are found for different values of π as follows:

n	QRm	n	QRm	n	QRm	n	QRm	n	QRm	n	QRm
1	45.0	2	26.34	3	18.26	4	14.2	5	11.10	6	9.98
1 $\frac{1}{2}$	38.40	2 $\frac{1}{2}$	23.58	3 $\frac{1}{2}$	16.54	4 $\frac{1}{2}$	13.15	5 $\frac{1}{2}$	10.47	7	8.8
1 $\frac{1}{4}$	33.41	2 $\frac{1}{4}$	21.48	3 $\frac{1}{4}$	15.57	4 $\frac{1}{4}$	12.32	5 $\frac{1}{4}$	10.1	8	7.8
1 $\frac{3}{4}$	29.45	2 $\frac{3}{4}$	19.50	3 $\frac{3}{4}$	14.56	4 $\frac{3}{4}$	11.58	5 $\frac{3}{4}$	9.52	9	6.20

N. B. The direction, or angle QRm, is to be taken below the plane for the suspending, and above the plane for the moving force.

Scholium. Though at first sight the former part of the above Problem, which shews the best method of applying an active force, seems superior to the other, yet, on farther consideration, the other appears of equal consequence, and particularly in building and fastening walls, banks of earth, fortifications, &c. and the application of what are called *land-ties*, &c. Thus if a weight, for instance, is to be drawn along the plane RB, and the friction be $\frac{1}{4}$ of the pressure, the best direction is when Rm makes an angle of 18°. 26' above the plane; but if the weight is a quantity of earth or stone, or any thing to be suspended, as in the case of land-ties, the best angle (on the foregoing supposition) must be 18°. 26' below the plane.

SCHOLIUM.

In those propositions the friction is estimated according to the most generally received opinion, that the resistance is proportional to the whole pressure compounded of the weight of the body, and the additional force necessary to overcome the friction; but it has been asserted, that there may be cases where the friction is not proportional to the whole pressure, but to that which would arise if the body was sustained in a given direction, exclusive of friction; and that there might also be cases, where the resistance, arising from tenacity or cohesion, might be as the relative pressure against the plane, and the force to overcome it the same in every direction; something similar to a globe stuck fast in wet tenacious clay: I shall therefore give solutions to both cases.

In the first case, (Fig. 307, Pl. 19.) the force requisite to sustain the body in direction RV, exclusive of friction, is Rn; and as Rn is equivalent to RD and Dn, therefore Ps is the pressure exclusive of

of friction; and as the friction is the n part of the pressure, the force acting parallel to AB to overcome it, is the n part of Pn ; but the force which acting in direction Rn will be equivalent to the n part of Pn in the direction Rn , is a fourth proportional to n times RD, Pn , and Rn ; but because DQ is the n part of DP, therefore fn is the n part of Pn , and the fourth proportional aforesaid will be nz ; consequently the sum or difference of Rn and nz must be a given quantity, or the least possible: the Problem therefore is reduced (Fig. 308, Pl. 19) to drawing a line Rn from the given point R, meeting the two lines PD and PQ given in position in n and z , so that nz added to or taken from Rn , the sum or difference may be a given quantity, or the least possible. To do this, let DS be taken equal to DR, and draw Sr parallel to PD meeting PQ in M; then because Rn is equal to rn , the sum or difference of the quantities aforesaid is rz ; and when rz is required to be a given quantity, the question is reduced to that particular case of the inclinations of Apollonius, in solids, which has been resolved by Newton and Barrow: the limits of the Problem, or the mode of drawing the line Rr , so that the intercepted part rz may be the least possible, may be investigated as follows:

(Fig. 310, Pl. 19.) Suppose it done, and Rrz the position required, and let Rnm be indefinitely near to Rz , and Mh perpendicular to Rz ; then by applying the analysis of the ancients to the *Newtonian* doctrine of prime and ultimate ratios, mn is equal to zr ; and if from the centre R, with the distances Rz and Rn , the arcs zv and nt be supposed to be described, vn is equal to zt , and consequently tr equal to mv ; but $rt : tn :: rh : Mh$, and $tn : zv :: Rr : Rz$, and $zv : vn :: Mh : hz$, whence by compounding the proportions, $tr : vn :: Rr . rh : Rz . zh$, and as the two first terms are equal, the two last are equal, and consequently $Rr : Rz :: zh : rh$, and dividing $Rr : rz :: zh : rz$, therefore Rr is equal to zh , and consequently the point h is in an hyperbola, whose asymptotes are QM and SM produced: but because the angle MhR is a right angle, the point h is also in the circumference of a circle; therefore a line drawn from R to h , the point where the hyperbola and circle intersect, is the position required.

In the other case, where the resistance arising from tenacity or cohesion is supposed to be as the relative pressure against the plane, and the force to overcome it the same in each direction, we have Rn for the sustaining force, exclusive of friction; and the n part of Pn for the friction; and consequently the sum or difference of these is the expression for the whole force; and the Problem may be thus constructed. Take PD to DQ as the pressure to the friction, and join PQ; on PD describe a circle, in which take Dv equal to DQ; join Pv, and draw RV perpendicular to it: then RV will represent

represent the direction and measure of the whole force when it is the least possible.

For DQ and Dv are equal, and consequently nf is equal to Vn ; but DQ is the n part of DP , therefore nf or Vn is the n part of Pn ; and consequently RV is equal to the sum or difference of Rn , and the n part of Pn : but RV is the least possible by construction, and therefore the other is a minimum also. For draw any other line Rk meeting RV in k and PD in m ; and draw mq , mt , parallel to DQ and Dv ; then the sum or difference of Rm and mt is equal to the sum or difference of Rm and mq ; but the sum or difference of Rm and mt is greater than RV , and therefore the sum or difference of Rn and the n part of Pn is the least possible.

PROBLEM III.

Given the weight of the body, the inclination of the plane, and the force sustaining the body in a given direction: to find the ratio of the friction to the pressure.

Take PR as before, (see Fig. 305, 306, Pl. 19.) draw Rm in the given direction, and take PR to Rm as the weight of the body to the force sustaining it; draw Pm meeting AB in Q , and PD perpendicular to AB ; then PD is to DQ as the pressure to the friction.

PROBLEM IV.

If $AhqN$ be the segment of an equilateral triangle, which, by moving parallel to itself and the horizon, generates a solid, upon which a figure $hmGEHKpqh$ moves, touching the former in hm and qp ; required the effect of the friction; still supposing it the n part of the pressure.

Let P be the centre of gravity of half the body (Fig. 311, Pl. 19.) and PR its weight as before; then the body by means of its inflexibility is kept together in the same manner as if it was actuated by a force parallel to the horizon; but if PDn be perpendicular to Ah , and Rn parallel to the horizontal line AC , meeting PD in n , Pn will be the pressure against the side Ah , and the friction is the n part of Pn ; but $PR : Pn :: AC : AB$; therefore if AC represent the weight of half the body, the n part of AB will express the weight requisite to overcome the friction for that half; and by doubling the expression they serve for the whole. Wherefore let W represent the weight of the body, f the secant of the angle BAC ; then Wf will be the pressure against the plane AD ; and the n part of Wf the force necessary to overcome the friction; and as this last is the force necessary to draw the body
along

along a horizontal plane, therefore the force necessary to draw the body along a horizontal plane, is to that necessary to draw it along the body whose section is $AhqN$, as AC to AB , or as 1 to f .

Because when the angle CAB is given, the ratio of PR to Pz is constant: therefore when the solid whose section is $AhqN$ is elevated, making an angle with the horizon, so that its base forms an inclined plane; PR in that case represents the pressure in a normal direction to that plane, and Pz the pressure against the solid; and as the friction is increased in the ratio of the pressure, therefore if the pressure which the body would have on the inclined plane be increased in the ratio of AC to AB , or radius to the secant of the angle CAB , then the pressure on the angular plane or body, whose perpendicular section is $AhqN$, will be had, and consequently its n part, or the friction. Hence this construction (Fig. 309, Pl. 19.); let PR represent the weight; then PD at right angles to AB represents the pressure that the body would exert against the common inclined plane; take DK to DP as AB in the foregoing figure to AC , or as the secant of the inclination of the angular plane with its base to radius; let Dq be the n part of DK , and join Kq ; then RM drawn any how to meet Kq in M , gives RM for the measure of the whole force in that direction; and it is the moving or suspending force, according as DQ is taken upwards or downwards in the line AB .

It is evident that Kq is parallel to PQ , and therefore though the least force (which is perpendicular to Kq) differ from that in the former cases; yet the directions for having the greatest effect are still the same as in the foregoing table; the demonstration is in effect the same as the first.

Corollary. By supposing f to be the secant of the angle (Fig. 312, Pl. 19.) that the sides of the angular plane make with the base, proceeding as Corollary 2d of Problem 1st, and putting t for the natural tangent of the plane's inclination, and W for PR the weight, we have $W (tn + f) : (n - t)$ for the moving; and $W (tn - f) : (n + t)$ for the suspending force, necessary to draw the body along the angular inclined plane by a force acting parallel to the base of the plane.

Example. Let AB , BC , and AC , be 5, 3, and 4, respectively, and let the inclination of the sides be 45° ; the weight of a ton and the friction one third of the pressure; then 3648 pounds is the moving, and 499 the suspending force.

SCHOLIUM.

In this proposition, those parts of the plane on which the body moves are supposed rectilinear, as mostly happens in practice; but the

the friction is easily estimated in curvilinear surfaces, and may be found generally as follows :

Let AMP (Fig. 313, Pl. 19.) be half the section perpendicular to the horizon, and to the axis of the solid which forms the curvilinear plane on which the body is moved; AP the axis: PM the ordinate, and MS a tangent to the curve at the point M; also let RM represent the weight or pressure in a direction perpendicular to the horizon at the point M; and let RF be perpendicular to MS meeting MP in F: also let PN be taken equal to MR, and PQ equal to RF; and suppose the same construction to be made for every point of the curve, and let HN be the locus of all the points N, and GQ the locus of all the points Q, then will the friction, when drawn along the horizontal plane, be to the friction of the same body when drawn along the curvilinear plane in the same direction, as the area APNH to the area APQG.

For the friction on the horizontal plane being as the sum of the pressures, is as the sum of all the elementary lines MR or PN; that is, as the area AHNP; and the friction on the curvilinear plane is for the same reason as the sum of all the RF or PQ, namely, as the area APQG; hence the truth of the proposition is manifest.

Corollary 1. Because Mx or the fluxion of y is to Mm the fluxion of the curve, as MR or PN to RF or PQ, therefore if PN be a function of AP, PQ will be a fourth proportional to the fluxion of the ordinate, the fluxion of the curve AM, and this function; wherefore if the curves HN and AM be given, the nature of the curve GQ will be known, and its area may be found by the common methods of quadratures.

Corollary 2. It is evident that when the planes are inclined to the horizon, the frictions of the right and curvilinear planes are still in the same ratio as in the preceding cases, and consequently may be found by the same mode of proceeding.

Corollary 3. It is also evident, that the above method holds good whether the parts of the body are connected together or not, with respect to their motion in the direction RM, so long as each elementary part MR may be considered as sustained at the point M by a force parallel to MP; but when the body is rigid or inflexible, the case becomes more simple, for MR is then constant, and APNH becomes a parallelogram.

Corollary 4. By supposing given properties to exist in any two of the curves AM, HN, or GQ, the nature of the third will be known; and hence a number of problems relative to friction may be proposed and resolved by a proper application of the direct and inverse methods of fluxions.

PROPOSITION 5. THEOREM.

In the application of forces to overcome friction, the same allowances must be made for the forces acting to advantage or disadvantage, by means of levers or other mechanical powers, as are made in the common doctrine; for instance, if a weight of two pounds, by acting at the distance of one foot from the fulcrum of a lever, be sufficient to overcome the friction, then one pound at two feet distance will have the same effect, &c.

This is two evident to need a demonstration.

OF FRICTION IN THE SCREW.

As any force acting perpendicular to the direction of a moving body does not affect the motion of the body in that direction, so the force acting perpendicular to the axis of the screw has no effect on the motion of a body raised thereby, exclusive of friction; it therefore requires the same force to raise a body by means of a screw, as to raise the same body in equal time along an inclined plane of the same elevation, as the threads of the screw by means of a force acting parallel to the base of the inclined plane: now, if we suppose the weight so contracted or condensed as to be capable of being placed on one of the threads of the screw, and fastened to an imaginary lever always perpendicular to its axis, then it is evident this lever will have no effect but to change the direction of the weight, and keep it in the midst of the thread of the screw; and if a force be applied at the weight always perpendicular to this lever, so as to sustain or draw it along, this force will be determined exactly the same as was done before in the inclined plane: but the rigidity of the parts of the "female screw" serves exactly the same purpose as this imaginary lever, and makes the weight act upon the threads like a body sustained on an inclined plane by a force parallel to its base; and as the force to overcome both the weight and the friction is reciprocally as the distance from the centre of the axis, therefore the distance of the power from the centre of the axis is to the distance the same centre to the middle of the threads of the screw, as the force necessary to sustain the body on the inclined plane, to the same force in the screw at the distance of the power. The same proportion holds good whether the threads be cut perpendicular to the axis or in an angle; for in the first, the common plane is to be taken; and in the second, the inclined or angular one, considered in the fourth Proposition: Wherefore if d be the distance from the centre of the axis to the middle of the threads of the screw, D the distance of same centre to the point where the force is applied, the force to overcome the weight and friction is Wd ($tn = f$): ($n = t$) D , where the letters

expres

express the same things as before, and the upper sign is for the moving, and the lower for the suspending force. N. B. t is the natural tangent of the angle made by a line touching one of the threads, and a plane at right angles to the axis of the screw; or it is equal to the distance of the respective edges of two threads, divided by the circumference of the cylinder, out of which the screw is cut.

Corollary 1. When lines drawn from the center of the axis of the screw to coincide with the threads, are at right angles to the axis, the above expression becomes $Wd (tn = 1) : (n = t) D$, for f becomes radius or unity.

Corollary 2. When n is equal to t , the moving force will be infinite; also the suspending force will be nothing when t is the part of f ; and when $Wd (tn = f) : (n + t) D$, becomes negative, it expresses the quantity of force, which must act in a contrary direction to reduce the body just to a state of suspension.

SCHOLIUM.

It would be needless to make any allowance for the curvilinear surfaces of the threads of screws, as they seldom differ much from the two foregoing forms; neither is it of much consequence to allow for their parts being at different distances from the axis, as their breadth seldom bears any considerable ratio to the length of the levers by which they act; but the case is different when large bodies revolve on each other, and therefore it will be necessary to shew the mode of proceeding in such cases.

Let $MmAQ$ be a convex solid, generated by the revolution of the curve MAQ about its axis perpendicular to the horizon, and $MRSQ$ a concave body exactly fitting it; then if this last body be revolved about the axis AP by means of the lever Pf , the force necessary to overcome the friction of one body turning upon the other may be found as follows. Suppose the revolving body divided into an infinite number of concentric tubes, that may descend independent of each other, and press freely against the body on which they revolve, and yet be so connected that the lever Pf may give the same angular velocity at the same time to each; also let the ordinates PN of the curve HN represent the weight or pressure (in a direction perpendicular to the horizon) of each of the indefinitely small parts Mt , or elementary lines of the body at the distance PM from the axis, and let c be the circumference of a circle whose radius is unity: then because the friction of each of the elementary tubes $MRSQ$ is as its pressure, and the pressure is as the number of lines Mt , and the pressure of each; therefore as this number is as $PM.Mt.c$, we have the n part of this expression

for the force which, acting at M , would overcome the friction of the cylindrical tube if moved round upon a horizontal plane; but as the pressure of each elementary part is increased in the ratio of Mn to Mm , when moved on the solid MAQ , the real force will be $(PM.c.Mm.PN) : n$; also $Pf : PM :: (PM.c.Mm.PN) : n$ to the small elementary force which will overcome the last force when acting at f ; consequently the whole force will be equal to the fluent of $(PM^2.PN.Mm.c) : (n, Pf.)$

Corollary. By means of the curves AM , HN , &c. conclusions may be drawn similar to those in the Corollaries to the Scholium of the fourth Proposition.

OF FRICTION IN THE LEVER.

It has been already observed, that a force acting perpendicular to the direction of a body in motion, does not alter the body's motion in that direction; therefore if (Fig. 315, Pl. 19.) we suppose DB to be an upright cylinder, and AB a body touching it in a line as in the figure, and retained close to it by an imaginary force, drawing it perpendicular towards the axis; then if a force CP be applied to C , the center of gravity of AB , and be always supposed to act perpendicularly to the radius CN , drawn from the centre of the axis to the point C , the friction will be the same in drawing the body round the cylinder, as in drawing it along a horizontal plane with an equal pressure; and if it be moved round by a force acting at a greater distance, the force will be reciprocally as the distance: on the contrary, if the body AB be fixed, and the cylinder turned round about its axis, the friction will be the same as if the cylinder was fixed, and the body drawn round it by CP , as before: Likewise the friction is the same, whether the cylinder be fixed, and the body AB moved round the axis MR by a force Qc applied at c ; or whether the point c be fixed with AB fastened to Cc , and the cylinder be revolved in a circle whose centre is c , so as always to retain its parallelism with respect to any fixt object; and as this last case obtains in the axletrees of carriages, since every point of the wheel's contact with the ground may be considered as the centre of motion for that instant, therefore the effect of the resistance arising from the friction of the concave part of the nave upon the axletree, is to the effect that would arise from drawing the same weight over a horizontal plane of the same kind, as the parts that rub each other, as the radius of the axis to the radius of the wheel. It must be observed, that this is not the only friction to which carriages are subject; for there is another part, arising from the cohesion of the wheel and the ground at their contact, which is to be found and allowed for by the three first Propositions.

In

In the above the pressure and friction have been supposed to be as the weight, as it is on a horizontal plane; but by the Scholium to the fourth Proposition, it is plain that the pressure is greater than the weight, and may be so in any proportion: however, as it appears by calculation, that the pressure on an arc of ninety degrees is to that on its chord, only as 1,183 to 1, when both the concave and convex parts have exactly the same curvature, the difference will be so trifling when the cylinders have different curvatures as usual, as to require very seldom to be allowed for.

This being premised, let M (Fig. 316, 317, 318, Pl. 19.) be a weight placed at the point A of a lever, moveable about an axis whose centre is d and radius dn ; and let N be the sustaining force acting at B : now it is evident that the pressure on the axis differs so little from the weight, that it may be safely taken for it without any considerable error, except in some remarkable cases, which may be allowed for from what has been said already; and therefore the friction which ought in strictness to be taken as the π part of the pressure, will here be taken as the π part of the weight upon the axis. Now if N be taken for the force which, acting at B , would be just sufficient to keep the weight M at A in equilibrio, exclusive of friction, and if W be the additional force to be added to N so as to overcome the friction; then will $M + P$, $M - P$, and $P - M$, be the weight upon the axis at d in the first, second, and third figures respectively, (supposing the sum of M and N to be equal to P). Now as the friction is the π part of each of these quantities, and its effect is to keep the lever in a state of rest, therefore in whatever direction the force at N endeavours to draw the lever by acting at B , the friction tends to counteract that force by keeping the lever steady, or acting in a contrary direction at π ; and as the effect of the friction, and the additional force W , are in equilibrio, and the friction acts by means of the lever dn , and the force W by the lever dB ; therefore Bd is to dn as the sum or difference of the π part of $N + W$ and M is to W ; consequently $W = dn (M + N) : (n.Bd - dn)$ in the first figure; $W = dn (M - N) : (n.Bd + dn)$ in the second figure; and in the third figure, $W = dn (N - M) : (n.Bd - dn)$. All these are the expressions for the moving forces.

To find the suspending forces, or the forces which, acting at N , shall be just sufficient to prevent the weight M from descending: Let M and N be the same as before, and let w be the force which, taken from N , will leave a force just sufficient to prevent M from descending; then the weight upon d in the first figure will be $M + N - w$; in the second figure, the weight will be $M - N + w$; and in the third figure, $N - M - w$; and by proceeding

as before, the values of w in the suspending forces are $dn (M + N) : (n.Bd + dn) : dn (M - N) : (n.Bd - dn)$, and $dn (N - M) : (n.Bd + dn)$, in the first, second, and third figures respectively.

Because $Bd : dA :: M : N$, therefore if this value of N be substituted in each of the above expressions for the friction, the whole force capable of sustaining the friction and weight M will be had. Thus, for example, the moving force to overcome the friction and weight M in the first figure, will be $M (n.dA + dn) : (n.Bd - dn)$, and the suspending force $M (n.dA - dn) : (n.Bd + dn)$; in the second figure, the moving force will be $M (n.dA + dn) : (n.Bd + dn)$, and the suspending force $M (n.dA - dn) : (n.Bd - dn)$; and in the third figure, the moving force will be $M (n.dA - dn) : n.Bd - dn$, and the suspending force will be $M (n.dA + dn) : (n.Bd + dn)$.

The method of finding n from each of the above equations is evident, and consequently the ratio of the friction to the pressure by experiments.

OF FRICTION IN THE WEDGE.

Let AC (Fig. 319, Pl. 19.) be the force necessary to sustain the wedge QPB in the direction aB perpendicular to QP , friction included; and let AB be the force exclusive of friction: draw AN and AH perpendicular to BQ and BP ; CG parallel to AN , and CF parallel to AH : Now GA and AF , the forces of the wood against the sides of the wedge, in those directions, compound a force equivalent to the diagonal CA in the direction CA , and therefore a force represented by AC in that direction, must be applied to the head of the wedge at a to overcome these forces. Let gr be the n part of Ag , and let the lines Ar be drawn, and also GK and FZ perpendicular to AG and AF , meeting the lines Ar in K and Z ; then will GK and FZ represent the friction against the sides BP and BQ , being each the n part of AG and AF , the pressure against each side respectively; wherefore if Be be taken in PB , and Bn in BQ , equal to GK and FZ respectively, the forces Be and Bn in those directions must compound a force to which the force BC in the direction BC must be equivalent; and consequently if Bm be the force compounded of Be and Bn , and Cm be joined, Cm must be perpendicular to mB ; since Be or GK is the force of friction arising from the pressure against BP , which tends to prevent the wedge from moving either in the direction BP or PB ; and Bn or FZ has a similar effect with respect to the direction in the line BQ ; and by hypothesis BC is just sufficient to balance these forces. It is also evident from what was said concerning the inclined plane, that Be and Bn must be taken in the directions PB and QB

QB for the moving force, but in the directions BP and BQ for the suspending force.

The method of calculation is evident; for as aB , AG , and AF , are perpendicular to QP , BP , and BQ , the triangles QPB and CAG are similar, and the parallelogram $Bnme$ similar to $FAGC$; whence by supposing certain parts given, the rest may be found, &c.

Corollary. When the wedge is isosceles the point m falls on C , and Be is equal to Bn , and therefore Be or GK is equal to $(AB + BC) PB : (n.QP)$; but $PB : Ba :: 2Be : BC$, and therefore $BC = 2Ba (AB + BC) : (n.QP)$ or equal to $(2Ba.BA) : (n.QP - 2Ba)$, and therefore $AC = (n.QP.AB) : (n.QP - 2Ba)$; and by following the same method for the suspending force, we find $BC = (2Ba.AB) : (n.QP + 2Ba)$ and consequently AC is equal to $(n.QP.AB) : (n.QP + 2Ba)$.

SCHOLIUM.

By proceeding in a similar method, the forces of the arch-stones of bridges may be determined; for let $QbbP$ be a stone sustained by the parts of the arch pressing against Pb and Qb , and let A be its centre of gravity, and AB perpendicular to the horizon; also let AB and AC be the same as before; then because the body is in equilibrio, the force in direction AC will be equivalent to the force in a contrary direction, arising from the pressures against the body in the directions GA and KA , together with the force of friction; and because the pressures are AG and AK , if Be , (the n part of AG) be drawn parallel to PB ; and Bn (the n part of AK) be drawn parallel to Qb ; and the parallelogram $Bnme$ be completed, and Cm joined; Bm will be the force arising from friction, and the angle BmC a right angle. The adjacent figure (Fig. 320, Pl. 19.) is for the moving force; but the method is similar for the suspensive force; and it is evident that the one construction is of use to determine the force which tends to break an arch by pressing it downwards, and the other the force that tends to break it upwards.

But as that excellent mathematician, *P. Frisi*, in his *Instituzioni di Meccanica*, has objected to the division of the force AB into the forces AN and AH , and thence concluded *Belidor* and *Couplet* to have been mistaken on that account in their writings upon bridges; I shall, therefore, prove that the common method is really in consequence of what that gentleman himself allows, and that his objections are not well founded. In the first place, he allows the force AB to be equivalent to the forces AV and AD or VB ; now (excluding friction) if that part of the arch which touches Pb was removed, it is evident $QbbP$ would immediately begin to descend

scend along Qb with a force represented by VB or AD ; but this descent is prevented by that part of the arch which touches Pb ; and therefore the force of that arch, in the direction HA , must be such as to be equivalent to DA in the direction DA or BV ; but no force greater or less than HA will be equivalent to DA in the direction DA , and therefore HA is the real pressure or force against Pb . Again, HD is the pressure in a perpendicular direction to Qb arising from this force; and as AV is the pressure against Qb arising from the force AB , therefore AV , together with HD , is the whole pressure against Qb in the direction AV ; but because the body is in equilibrio, and consequently the action or force in the direction AV equal to the reaction in a contrary direction; therefore $AV + HD$ or AN (because NV is equal to HD by the property of the parallelogram) represents the pressure against Qb , and AH the pressure against Pb ; which is contrary to what *P. Frisi* asserts, and agreeable to the usual method.

The same learned Author has made another very material mistake, from a similar cause, at page 67 of the aforesaid Treatise, relative to the tension of ropes; which cannot be attributed to haste or inadvertency, as he expressly asserts the holders of the common opinion to be mistaken, in consequence of their using the theory of composition of forces without sufficient precaution: I shall, therefore, after giving his own words, take the liberty of shewing where I apprehend he is mistaken.

“ Parleremo più a lungo delle altre ricerche matematiche, alle quali ha dato occasione la controversia insorta intorno alla cupola di S. Pietro. Coll' occasione che si è discorso in Milano di munire la fabbrica del Duomo di un Conduttore elettrico, che dalla cima dell' aguglia si dirimasse, e scendesse per differenti parti del tempio, si è ancora parlato dell' azione, che i fili del Conduttore protrebbero esercitare contra l'aguglia, e si sono proposti varj Problemi intorno alle tensioni delle funi. Io qui aggiugnerò le soluzioni, che ho ritrovati, e incomincerò pella prima risoluzione delle forze tendenti, la quale siccome è interamente differente da quella che hanno seguitato altri Autori, così non sarà meraviglia che porti dei risultati interamente differenti da quelli che sono stati finora publicati. Penda il (Fig. 321, Pl. 19,) filo QVR , dai punti Q , ed R , e vi si attacchi in V il peso P . si produca la verticale PV in A ; si esprima il peso P colla retta AV , e dal punto A ; si tirino sopra QV , RV le perpendicolari AM , AN . Sarà MV l'intera forza esercitata secondo QV , ed NV sarà quella che si eserciterà secondo RV .

“ La stessa cosa si dedurrebbe risolvendo la forza AV nelle due Aq , Ar parallele ai fili QV , RV , e poi risolvendo di nuovo la forza Aq nelle due AN , Nq , e similmente la Ar in due altre AM ,
Mr.

Mr. Mentre queste risoluzioni è manifesto che la forza totale esercitata nel tendere il filo QV dev'essere $Aq - Mr = rV - Mr = MV$, e la tensione del filo $RV = Vb - Nq = NV$.

“ S'ingannerebbe chi misurasse separatamente la tensione del filo QV dalla forza Aq , ossia rV , e la tensione di RV da Ar , oppure da qV . Egli è vero, che le due tensioni equivalgono insieme, come alla sola forza AV, così ancora alle due Ar , Aq , oppure alle quattro insieme AN, Nq , AM, Mr. ma nel prendere le tensioni separate bisogna in oltre avvertire, che quando l'angolo QVR non è retto, una porzione di Aq agisce secondo RV, ed una porzione di Ar secondo QV: e separando le azioni sarà MV, la tensione del filo QV, ed NV quella di RV.”

In the first place, I shall demonstrate the truth of the established method from principles that FRISI has himself allowed; and, secondly, point out the absurdity of his conclusions.

1. Let Vn and Sr be parallel to AN; then because NVn is a right angle, and the force VA may be resolved into VN and Vn , in those directions, therefore, if RV and VP were to remain in the same position, and the force which now keeps the body suspended by acting in the direction VQ, was to act in the direction Vn with a force expressed by Vn , it is then granted that the equilibrium would still be maintained, and the tensions would be as Vn and VN; and therefore as no force VS whatever acting at V in the direction RV, can have any effect in the direction Vn perpendicular to RV, it necessarily follows, that the force in any other direction VQ must be such as to be equivalent to Vn in the direction Vn ; but it is likewise granted, that no other force but Vr in the direction VQ can be equivalent to Vn in the direction Vn ; and as the force Vr is equivalent to Vn and VS, and as VS, or its equal, qN , only gives an additional tension to NV, the tension which the chord RV was supposed to have before, which whole tension is equal to the reaction of the tack R; therefore qV is the tension of the chord RV, and Vr that of Qv.

2. Let the points Q and R coincide, and RV, QV, and VP, will then be perpendicular to the horizon; and if VQ or VR be assumed to express the weight P, then will the point A, R, Q, M and N coincide: and according to *Frisi's* principle, the tensions of RV, VQ, and VP, will be equal; but, from the well known principle of the pulley, each chord VQ and VR bears but half the weight P, and therefore this absurdity follows, that a chord is as much stretched with half the weight as it would with the whole.

Again, if the points R, V and Q be supposed horizontal, it follows, from the common theory, that the tension of the rope RVQ would be infinite; but VN and VM vanish when RVQ is horizontal; and therefore, by *Frisi's* principle, the tension in that
case

case would be nothing at all; but it is well known from the most common experiments to be very considerable, even when RQV is but nearly horizontal; and therefore the new theory of this great mathematician is indefensible.

Remark. All the foregoing, except the last Scholium, was written in 1775, before the author had seen any thing to speak of on the subject. He had designed and executed great part of an extensive treatise on friction, according to different hypotheses; but as no body would be at the risk of publishing it, and he could not afford it himself, the most of it was accidentally lost. What is here given is an extract only of some of the first part, where velocity was not taken into the account, and where there were no complicated algebraic or fluxional expressions, which would be difficult to print in this country.

N. B. This curious paper is taken from the first vol. of Asiatic Researches.

ARTICLE XXVIII.

FOUR PROPOSITIONS, &c.

SHEWING NOT ONLY THAT THE DISTANCE OF THE
SUN,

As attempted to be determined from

THE THEORY OF GRAVITY,

BY A LATE AUTHOR,

*Is, upon his Own Principles, Erroneous, but also, that it is more than
probable*

THIS IMPORTANT QUESTION

CAN NEVER BE SATISFACTORILY ANSWERED BY ANY
CALCULUS OF THE KIND.

SEVERAL of our Readers, it may be presumed, will recognize the following Propositions, and the circumstances from which they originated; as there are others who will not, we subjoin, for their information, the following particulars.

In the year 1761, Dr. MATTHEW STEWART, Professor of Mathematics in the University of Edinburgh, published a Volume
of

of Tracts, *Mathematical and Physical*, in which he proposed to determine the Sun's distance from the Earth, by a calculation founded on the Theory of Gravity. Two years afterwards the calculation appeared, in his Essay on *The Sun's Distance*, and the result so far exceeded all former computations, as to produce attentive enquiries into the principles he had adopted. His method though highly ingenious, and reflecting honour on his Talents, was at length found to be liable to many inconveniences, and inadequate to the purpose. To point out some instances of its failure, these PROPOSITIONS were published, anonymously, in 1768, by Mr. DAWSON, then Surveyor, at Sudbury, in Yorkshire, in which his candour and abilities are equally conspicuous. The Work is now so scarce as to be almost unattainable; and, as the subject still retains its importance, we are happy in being able, through the medium of some Mathematical Friends, and the permission of the Author, to lay its contents before our Readers.

PROPOSITION I. *Fig. 278, Pl. 18.*

Supposing the orbit of the moon to be a circle, and to coincide with the plane of the ecliptic; it is required to determine the forces with which the sun disturbs the motion of the moon round the earth.

Let M represent the moon in its orbit ABCD, E the earth, and S the sun. Draw SM and produce it to Q, so that SE may be to SQ as SM^2 to SE^2 . Then, if SE represent the gravity of the earth to the sun, SQ will represent the gravity of the moon to the sun when at M. From the point Q draw QN parallel to ME, meeting the line SE produced in N. Likewise draw ML parallel to EN, LE from the point L to the centre E, and LO perpendicular to ME. The force of the moon to the sun, represented by QS, may be resolved into the two forces QN, NS, and in those directions. But the line SE represents the force of the earth to the sun, which if taken out of the line SN leaves NE. Hence QN and NE are the two forces which disturb the moon
in

in its progress round the earth; the joint effect of which in increasing or decreasing the gravity of the moon towards the earth in any part of its orbit, as M, is what we are here enquiring after.

From M draw MG perpendicular to EC, and put $SE = a$, CE or $ME = r$, EG or $MF = y$; then, from the above proportion, $SQ = \frac{SE^2 \times SE}{SM^2} =$ (by substitution) $\frac{a^3}{a^2 - 2ay + r^2}$.

From SQ take $SM = \sqrt{a^2 - 2ay + r^2} = a - y$ nearly, and there remains $MQ = \frac{a^3}{a^2 - 2ay + r^2} - \frac{a^3}{a^2 - 2ay + r^2} = \frac{3a^2y - 2ay^2 - ar^2 + r^3y}{a^2 - 2ay + r^2}$. If r^3y be rejected in the numerator, and r^2 in the denominator, the value of the fraction will be very near the same as before, (both numerator and denominator being decreased at the same time, and the quantities rejected being much less than the rest) that is, MQ will then be $= \frac{3ay - 2y^2 - r^2}{a - 2y^2}$ nearly; which expression let us put $= z$,

The triangles SME, MQL, are similar; hence $SM = \sqrt{a^2 - 2ay + r^2} = a - y : SE = a :: MQ = z : ML = \frac{az}{a - y}$; and $SE = a : ME = r :: ML = \frac{az}{a - y} : QL = \frac{rz}{a - y}$. That part of the disturbing force represented by QL acts in the direction QL or ME from M towards E, and consequently increases the gravity of the moon towards the earth.—The force MQ may be resolved into the two forces QL and

and LM, and in those directions. QL acts directly towards the earth, and therefore need not be resolved into any other forces. The force ML and the remaining part LN are reduced into LE, and in that direction. This, again, is resolved into LO, OE, and in these directions. Now the force LO is exactly balanced when the moon is at a point equidistant from C on the other side of its orbit, and therefore, taking a whole revolution together, does not disturb its motion round the earth. The force OE therefore is what we are to estimate.

From the similar triangles LOM, EFM, we have

$$MO = \frac{ML \times MF}{ME} = \frac{ayz}{r \times a - y}, \text{ therefore, } OE$$

$$= \frac{ayz}{r \times a - y} - r = (\text{by substituting for } z \text{ its value})$$

$$\frac{3a^2y^2 - 2ay^3 - ayr^2}{r \times a^2 - 3ay + 2y^2} - r = (\text{by reduction}) \text{ to}$$

$$\frac{3a^2y^2 - 2ay^3 - a^2r^2 + 2ar^2y - 2r^2y^2}{r \times a^2 - 3ay + 2y^2}.$$

This force is in the direction EM, and consequently lessens the gravity of the moon to the earth. Hence

$$\text{if the force } QL = \frac{rz}{a - y} = (\text{by substitution})$$

$$\frac{3r^2ay - 2r^2y^2 - r^4}{r \times a^2 - 3ay + 2y^2} \text{ be taken from the above, there}$$

$$\text{will then remain } \frac{3a^2y^2 - a^2r^2 - 2ay^3 - r^2ay + r^4}{r \times a^2 - 3ay + 2y^2};$$

an expression for the force that must be deducted from the gravity of the moon to the earth, in order to obtain the true force with which the moon is im-

pelled towards the earth when at M. If r^4 in the numerator, and $2y^3$ in the denominator be neglected, and a struck out of every term, we have

$$\frac{3ay^2 - ar^2 - 2y^3 - r^2y}{r \times a - 3y} = \text{the true disturbing force}$$

nearly. (For besides the terms neglected bearing but a very small proportion to the rest, both numerator and denominator are diminished at the same time, consequently the value of the fraction must be very little altered). Again, if $2y^3$ and r^2y be likewise neglected, the expression for the disturbing

$$\text{force becomes } \frac{3ay^2 - ar^2}{r \times a - 3y}; \text{ which, though less ac-}$$

curate than the above, is much more simple, and turns out the very same that professor *Stewart* has found prop. 6th, tract 4th, of his *Traacts Physicall and Mathematical*, as will easily appear, by introducing our symbols into his conclusions, &c.

COROLLARY I.

When the moon comes to K, the other end of the diameter MK, the disturbing force will become

$$\frac{3ay^2 - ar^2 + 2y^3 + r^2y}{r \times a + 3y}; y \text{ being here negative.}$$

Hence half the sum of these two forces will be a

$$\text{mean solar one, and } \frac{3a^2y^2 - a^2r^2 - 6y^4 - 3r^2y^2}{r \times a^2 - 9y^2}.$$

And if Dr. *Stewart's* expression, found above, be reduced

reduced in like manner, we shall have $\frac{a^2}{r} \times \frac{3y^2 - r^2}{a^2 - 9y^2}$, for the mean force; which is the same as that he has determined prop. 8th of the before-mentioned tract.

COROLLARY II.

The proportion between the mean solar force determined above, and the force of the moon to the earth, is easily determined. Let p represent the periodic time of the moon round the earth, P the periodic time of the earth about the sun; then, the force of the earth to the sun, is to the mean solar force, as a is to $\frac{3a^2y^2 - a^2r^2 - 6y^4 - 3r^2y^2}{r \times a^2 - 9y^2}$. And

(by the laws of central forces) the force of the moon to the earth, is to the force of the earth to the sun, as $\frac{r}{p^2}$ to $\frac{a}{P^2}$: Hence the force of the moon to the

earth, is to the mean solar force, as $\frac{r}{p^2}$ is to $\frac{1}{P^2} \times \frac{3a^2y^2 - a^2r^2 - 6y^4 - 3r^2y^2}{r \times a^2 - 9y^2}$, or, as r to $\frac{p^2}{P^2} \times$

$$\frac{3a^2y^2 - a^2r^2 - 6y^4 - 3r^2y^2}{r \times a^2 - 9y^2}.$$

PROPOSITION II. *Fig. 278, Pl. 18.*

Required to determine the proportion of the variation of the sun's distance from the earth, to a corresponding

ponding variation of the disturbing force of the sun upon the moon at any point M in its orbit.

It is proved by the writers on this part of physical astronomy, that the distance of the moon from the earth increases and decreases as the distance of the sun decreases and increases; but an indefinitely small variation of the sun's distance, will scarce at all effect the distance of the moon. Therefore, in the expression for the disturbing force, viz.* $\frac{p^2}{p^2} \times$

$$\frac{3ay^2 - ar^2 - 2y^3 - r^2y}{r \times a - 3y}, r \text{ may be looked upon as a}$$

a constant quantity, and consequently, y will likewise be given for any given point M in its orbit. If, therefore, the expression for the disturbing force be thrown into fluxions, supposing only a variable, we shall have the variation of the disturbing force =

$$\frac{p^2}{p^2} \times \frac{a \times -y}{a - 3y|^2} \times \frac{7y^2 - 4r^2}{r} : \text{ But the variation of}$$

the sun's distance is \dot{a} ; hence the variation of the disturbing force is to the variation of the sun's distance,

$$\text{as } \frac{p^2}{p^2} \times -y \times \frac{7y^2 - 4r^2}{r} \text{ to } \overline{a - 3y|^2}.$$

If $\frac{p^2}{p^2} \times \frac{n}{r \times a - 3y}$ be neglected from the disturb-

ing force, n being supposed an indefinitely small negative or positive quantity) the variation in the sun's distance, occasioned thereby, will be found by this proportion, viz. as $\frac{p^2}{p^2} \times -y \times \frac{7y^2 - 4r^2}{r}$ is to $r \times \overline{a - 3y|^2}$,

* This will easily appear from Cor. 2. last Prop.

$a - 3y$ is, so is $\frac{p^2}{p^2} \times \frac{n}{r \times a - 3y}$ to $a - 3y \times$

$\frac{n}{-y \times 7y^2 - 4r^2} =$ the variation in the sun's distance, consequent upon neglecting an indefinitely small quantity (n) in the numerator of the fraction, expressing the disturbing force.

COROLLARY I.

For any given point in the orbit, the variation in the sun's distance is as $a - 3y \times n$; that is, as the sun's distance multiplied into the part neglected in the numerator, nearly. This is evident from what is gone before. If n be supposed constant, the variation will then be as $a - 3y$, that is, directly as the distance, nearly; for $3y$ is very small compared with a .

COROLLARY II.

Let us now suppose $\frac{p^2}{p^2} \times n$ to be any quantity neglected from the expression for the disturbing force; then proportioning, as above, there comes out

$a - 3y \times \frac{rn}{-y \times 7y^2 - 4r^2}$ for a corresponding varia-

tion of the sun's distance; which is as the quantity neglected multiplied into the square of the distance, nearly; but if n be constant, it is as the square of the distance directly; for $3y$, as was observed before, bears only a very small proportion to a .

SCHOLIUM.

From the last proposition and its corollaries it will plainly appear, that the variation of the disturbing force is extremely small, when compared with a corresponding variation of the sun's distance, and this still less as the distance of the sun is greater; a conclusion directly contrary to the opinion of Dr. STEWART. For, in the preface to *Traacts Physical and Mathematical*, we have the following paragraph: "It is well known that the various methods hitherto attempted to solve this curious problem, have failed in a great measure, on account of the vast distance of the sun from the earth; but the method hinted at here, will give the solution the more accurately, the more distant the sun is from the earth; for it proceeds on the supposition that the distance of the sun from the earth is great."

Upon a slight view of the matter this assertion looks probable. For, if any quantity (n) be neglected in the numerator of the fraction expressing the dis-

turbing force, its value in the fraction is as $\frac{n}{r \times a - xy}$,

which is evidently less as a , or the distance of the sun from the earth, is greater: And, therefore, it would seem to follow, that it would less affect the distance of the sun, determined by this method. But this, as we have found above, is just the reverse: And here let us consider, whether the single truth just demonstrated, does not of itself sufficiently prove, that the distance of the sun can scarcely ever be truly ascertained by the *theory of gravity*.

PROPOSITION III. *Fig. 323, Pl. 20.*

the mean solar force for any point of the orbit, as M,
being

being $\frac{p^2}{P^2} \times \frac{3a^2y^2 - a^2r^2 - 6y^4 - 3r^2y^2}{r \times a^2 - 9y^2}$, (cor. 2, prop. 1); it is required to find the sum of all the mean forces for a quadrant of the orbit BC.

Draw the line bc , equal to the arch BC, and let m , in the line, correspond to M, in the arch. At m erect a perpendicular, and therein take mr equal to the mean disturbing force at M $= \frac{p^2}{P^2} \times \frac{3a^2y^2 - a^2r^2 - 6y^4 - 3r^2y^2}{r \times a^2 - 9y^2}$. Let this be done for every point of the line bc , and draw the curve $qr sn o$ bounding these perpendiculars.

From this construction it may easily be seen, that the curve will cut the axis bc in some as n , that is, when $3a^2y^2 = a^2r^2 + 6y^4 + 3r^2y^2$; for then the ordinate $mr = 0$. From n to b the ordinates are all negative, and from n to c all positive.

Suppose y to be a flowing quantity, and all the rest constant, the fluxion of the arch BM, or line $bm =$

$$\frac{r\dot{y}}{\sqrt{r^2 - y^2}} = \frac{\dot{y}}{\sqrt{1 - y^2}}, \text{ (putting } r = 1 \text{); then } \frac{\dot{y}}{\sqrt{1 - y^2}} \times \frac{p^2}{P^2} \times \frac{3a^2y^2 - a^2 - 6y^4 - 3y^2}{1 \times a^2 - 9y^2} = \text{fluxion of the}$$

area of the curve-lined space $obcq$. The expression for the ordinate being reduced into a series, we have

$$mr = \frac{p^2}{P^2} \times \frac{-1 + \frac{3a^2 - 12}{a^2}y^2 + \frac{21a^2 - 108}{a^4}y^4 + \frac{189}{a^6}y^6 \&c.}{1}$$

Therefore.

Therefore, the fluxion of the space $= \frac{\dot{y}}{\sqrt{1-y^2}} \times$

$$\frac{p^2}{p^2} \times -1 + \frac{3a^2-12}{a^2} y^2 + \frac{21a^2-108}{a^4} y^4 + \frac{189}{a^4} y^6 \&c.$$

and the fluent, when $y = 1$, becomes $\frac{p^2}{p^2} \times$

$$-c + \frac{c}{2} \times \frac{3a^2-12}{a^2} + \frac{3c}{8} \times \frac{21a^2-108}{a^4} + \frac{5c}{16} \times \frac{189}{a^4} \&c.$$

c being put $=$ line bc or arch $BC = \frac{3.14159}{2}$. This

expression is equal to the difference of the two areas obn , and cnq , or $= cnq - obn$; that is, equal the sum of the positive forces, that have acted upon the body in passing from B to C , made less by the negative forces.

COROLLARY I.

Draw the line tp parallel to bc , and at such a distance from it, that the area $tbc p$ may be $=$ the area $cnq - obn$; then will tb or $cp = \frac{p^2}{p^2} \times$

$$-1 + \frac{1}{2} \times \frac{3a^2-12}{a^2} + \frac{3}{8} \times \frac{21a^2-108}{a^4} + \frac{5}{16} \times \frac{189}{a^4} \&c.$$

For the base c , multiplied into this perpendicular, gives the area of the parallelogram $tbc p = \frac{p^2}{p^2} \times$

$$-c + \frac{c}{2} \times \frac{3a^2-12}{a^2} \&c. = \text{the curve lined area}$$

found

found above. This perpendicular is therefore a *mean solar force*, which, when reduced, becomes ==

$$\frac{p^2}{P^2} \times \frac{8a^4 + 30a^2 + 297}{16a^4}, \text{ } r \text{ being all along supposed equal 1.}$$

COROLLARY II.

If the mean disturbing force for the point M, made use of by *Professor Stewart*, be taken equal the ordinate *mr*, and the curve lined area, found and reduced in the same manner as above, we shall

$$\text{have the mean solar force} = \frac{p^2}{P^2} \times \frac{8a^4 + 90a^2 + 729}{16a^4}$$

SCHOLIUM.

As *Dr. Stewart* makes use of the mean solar force found above, for determining the sun's distance, it will not be amiss to examine what proportion a small variation in the sun's distance bears to a corresponding one in the mean solar force; and likewise, whether a mean force can properly be made use of in the enquiry at all, or not.

Let the variation of the sun's distance be represented by *a*, then the variation of the mean solar force, viz. $\frac{p^2}{P^2} \times \frac{8a^4 + 30a^2 + 297}{16a^4}$, will be found,

$$\text{when thrown into fluxions,} = \frac{p^2}{P^2} \times \frac{-15a^3a - 297a}{4a^4}$$

Therefore, the variation of the sun's distance is to the variation of the mean solar force, as $4a^5 \times P^2$ to $p^2 \times -15a^2 + 297$, or (by neglecting 297) as $4a^5$

But to proceed with another objection. Since the perpendicular *cp* (fig. 323, pl. 20) is plainly an arithmetical mean among all the perpendiculars which may be conceived erected upon the line *bc*, the mean disturbing force, which it represents, must be an *arithmetical mean* among all the forces, which have acted in the description of the quadrant. Now, were the motion of the apses directly in the simple ratio of the disturbing force, then if any number of disturbing forces, all different from one another, were to act each an equal time, it is evident they would occasion the same motion in the apses, as if a force, which was an arithmetical mean among these forces, were to act for the whole time. But it will presently be seen, that the motion of the apses is not in the direct simple ratio of the disturbing force: hence it must be manifest, that a mean solar force, of the nature of the above, can never rightly answer the purpose for which it is intended, were it ever so accurately determined.

(To be continued.)



ARTICLE XXXI.

USEFUL PROPOSITIONS IN GEOMETRY.

By Mr. M. A. HARRISON.

(Continued from page 25.)

PROP. XIII. THEO. Fig. 257, Pl. 17.

IF *RS* be produced to meet the circle at *m*, and the points *C, m; F, m*; be joined; *Cm*, will pass through the centre *O*, and *Fm*, will cut *IB* at right angles in *L*.

Demon.

Demon. Since the points C, R, m , are in the periphery, and the angle CRm , a right one, it is evident that Cm is the diameter, and consequently passes through the centre O .

Again, since the arch CF is equal to the arch Em , and the points E , and F , being in the periphery, the angles FEC, EFm , standing on equal arches, will be equal, and therefore Fm will be parallel to EC ; but it has been shewn that CE is perpendicular to IB , therefore Fm is perpendicular to IB .

Q. E. D.

PROP. XIV. THEO.

Let K' be the point where a perpendicular to LT , from L , meets TU . Then if $L'K'$ be joined, it will pass through L , the middle of the base.

Demon. Join $L'L, K'L$, and let FH , drawn \perp to AC , meet CE in V , and join LH . It has been shewn,

that FI, PH ; and FC, IG , are respectively \equiv and paral. therefore FH is parallel and equal to IP ; hence LT or PG is equal to HC .

Now, since LT is equal and parallel to HC , the lines TC, LH , (joining their extremes) will be \equiv and \parallel : but $TC, L'F$, are equal and parallel;

therefore $LH, L'F$, are equal and parallel, and therefore $HF, L'L$, are equal and parallel.

Hence $L'L$ is perpendicular to LT , that is, the points L', L, K' , are in the same straight line. *Q. E. D.*

Cor. The triangles, $FCV, L'TK'$; $FCH, L'TL$; $VCH, K'TL$, are respectively equal in all respects.

PROP. XV. THEO.

Lines being drawn as before, FC ($\equiv ER \equiv Em$), will be equal to the diameter of the circle passing through the points L, T, D .

Vol. II.

X.

Demon.

Demon. Produce $L'K'$ to meet the circle ULTD in T' , and join TT' . Since TLT' is a right angle, and the points T, L, T' , in the periphery, it is evident that TT' is the diameter of the circle, whose centre is o .

Again, since Lo is \parallel to $L'T$, the $\angle s$ $T'Lo, T'L'T$ will be equal; but Lo is $= oT'$; hence, the \angle $LT'o = T'Lo = T'LT'$; theref. the $\triangle T'LT'$ is isosceles, and conseq. $L'T = TT'$; but $L'T = FC$; therefore $FC = TT'$, the diameter.

Q. E. D.

Otherwise. Because oL is parallel to TL' ,

$$T'o : T'T :: oL : TL'$$

but the antecedents $T'o, oL$ are equal;
therefore the consequents $T'T, TL'$ are equal;
but $TL' = FC$; therefore $T'T = FC$. *Q. E. D.*

Cor. 1. $L'L = LT'$, and $L'K' - K'T' = 2 LK'$.

Cor. 2. $T'L$, produced, will meet AC at right $\angle s$ in P .

PROP. XVI. THEO.

Things remaining as before, I say the perpendicular on , will be equal to half NF , or half SE .

Demon. Lo will evidently meet the circle in the point C' , where the circle cuts CDR .

Since $LC' (TT') = FC$, and $LD = NC$; DC' will be $= NF$, because the angles at N and D are right ones:

but $Ln = nD$, and no is parallel to DC' ;
therefore, $no = \frac{1}{2} DC' = \frac{1}{2} NF$, or $\frac{1}{2} SE$. *Q. E. D.*

Cor. 1. Join Do , and produce it to meet the circle in A' , and the points L, A', E , will be in a right line.

Cor. 2. LA' is equal to DC' .

PROP. XVII. THEO.

Things remaining still as before, I say the triangles ABC, TDU , will be to one another in the constant ratio of FE to FN .

Demon.

Demon. Since the Δ s ACB, TDU, are naturally equiangular, their areas will be to one another as the squares of their homologous sides, that is, as Δ ACB : Δ TDU :: AB^2 : TU^2 :: LB^2 : TQ^2 (QU^2) :: EB^2 ($EF \cdot EL$) : LT^2 ($ES \cdot EL = NF \cdot EL$, by Prop. IV.) :: FE : FN . Q. E. D.

Cor. When EF is a given line, and the Δ TDU, either a given space, or a maximum, the solid FNL · LB will either be a given solid, or a maximum.

ARTICLE XXXII.

ATWOOD'S INVESTIGATIONS ON WATCH BALANCES.

(Continued from page 136.)

BUT if the points of quiescence of the balance and auxiliary springs, instead of coinciding, according to the principle of Mr. MUDGE's construction, should deviate from this adjustment by a small arc $OQ = ON$ (fig. 82 and 83, pl. 5.); to what extent the daily rate of the time-keeper may be affected by this alteration in the position of the quiescent points, remains unknown, unless it be investigated by assuming the deviation of the points of quiescence, as one of the conditions on which the time of vibration depends. This condition is included in the preceding investigations, which may be now applied to the solution of some cases, which are suggested by considering the construction of Mr. MUDGE's time-keeper.

Any minute or particular description of this ingenious invention would be foreign to the subject of theoretic investigation; an outline only of the construction on which the action of the several springs

depends will be necessary, to render the application of the preceding theorems sufficiently intelligible.

ONEBQ (fig. 86, pl. 6.) is the circumference of the balance, vibrating by the action of a spiral spring on an axis CADH, passing through the centre C; the axis is discontinued from A to D to make room for the other parts of the work. CA and DH are connected by means of a branch or crank AXYD, which is fixed to the axis CADH, and always vibrates with the balance on the said axis.*

LM, ZW, are two rods affixed to the crank at the points L and Z parallel to XY; which rods also vibrate with the balance. *c, d, e, f, g, r, s,* are fixed parts of the machine. TR is an axis in the same right line with CADH carrying an arm GO at right angles to it † (or nearly so), and a small auxiliary spring *u*, which is wound up whenever the arm GO is turned round the axis TR in the direction of the arc Oh; *p* is a curved pallet fixed to the axis TR, which receives the tooth of the balance-wheel near the axis; the tooth proceeding along the curved surface by the force of the main spring, turns the axis and the annexed arm GO in the direction of the arc Oh, and at the same time winds up the auxiliary spring *u*. A small projection at the extremity of the curved surface of the pallet *p* prevents further progress of the tooth, when the arm OG has been turned through an arc OAh ‡ of about 27° ; consequently the spring *u* has then been wound through the same arc or angle $OGh = 27^{\circ}$.

FS is another axis in the same right line with

* The additional weight affixed to the balance at *e* counterpoises the weight of the crank or branch AXYD, so as to bring the centre of gravity of the whole into the axis of motion.

† In the machine GO and IO are not exactly at right angles to TR and FS; but are so represented in the figure, in order to make the different positions of the arms GO and IO the more distinct.

‡ *Vide infra* the note in page 240.

CADH,

CADH, exactly similar to that which has been described. This axis **FS** carries with it the arm **IO**; the auxiliary spring *v* is annexed to this axis, and is wound up when the axis is turned in the direction of the arc **Oh**. *q* is a pallet similar to the former pallet *p*, and is placed so as to receive the tooth of the balance-wheel, which by its action on the pallet winds up the spring *v*, and carries the arm **OI** through an angle **OIk**, $= 27^\circ$, further motion being prevented by a small projection at the extremity of the pallet *q*, similar to that which has been already mentioned. *lm* represents the balance-wheel, the upper tooth of which acts on the pallet *p*, and the lower tooth on the pallet *q*, alternately winding up the auxiliary springs *u* and *v*, in the manner described in the subsequent page: the axis of the balance-wheel *no* is parallel to the line **CO** or **GQ**.

The several arcs expressed on the circumference of the balance, i. e. **OQ**, **Oh**, **ON**, **Ok**, are equal to the respective arcs denoted by the same letters in the circumference of the circles described by the extremities of the arms **GO** and **IO**.

Suppose that, when the balance is quiescent, the main-spring being unwound, the branch or crank **AXYD** is in the position represented in fig. 86, **AX** being parallel to **CO**; if the quiescent points of the auxiliary springs coincide with that of the balance spring, the arm **GO** will just touch the rod **LM**; and in like manner the arm **IO** will just touch the rod **WZ**; the two arms **GO** and **IO**, in this position, are parallel to the line **CO**. This position of the balance and auxiliary springs remains as long as the main-spring of the machine continues unwound; but whenever the action of the main-spring sets the balance-wheel in motion, a tooth thereof meeting with one or other of the pallets *p* or *q*, will wind up one of the auxiliary springs; suppose it should be the auxiliary spring *u*: the arm **GO** being carried into the position

Gh , by the force of the balance-wheel acting on the pallet p , remains in that position as long as the tooth of the balance-wheel continues locked by the projection at the extremity of the pallet p : and the balance itself, not being at all affected by the motion of the arm GO , nor by the winding up of the spring u , remains in its quiescent position; consequently no vibration can take place except by the assistance of some external force to set the machine in motion. Suppose an impulse to be given to the balance sufficient to carry it through the semi-arc OB , which is about $\ast 135^\circ$, according to Mr. MUDGE's construction.

The balance during this motion carries with it the crank $AXYD$, and the affixed rods LM , ZW . When the balance has described an angle of about $27^\circ =$ to the angle OCh or OGh , the rod LM meets with the arm Gh , and by turning the axis TR , and the pallet p , in the direction of the arc Oh , releases the tooth of the balance-wheel from the projection at the extremity of the pallet p : the balance-wheel immediately revolves, and the lower tooth meeting with the pallet q , winds up the auxiliary spring v , and carries the arm IO with a circular motion through the angle OIk about 27° ; in which position the arm IO remains as long as the tooth of the balance-wheel is locked by the pallet q . While the spring v is winding up through the arc Ok , the balance describes the remaining part of the semi-arc kB , and during this motion the rod LM carries round the arm Gh , causing it to describe an angle kCB , or kGB \dagger , which is measured by the arc $kB = \dagger 108^\circ$: when the balance has

** Vide infra the note in page 240.*

\dagger The angle kGB is not expressed in the figure for want of room, but is easily imagined, being precisely equal to the angle kCB represented on the balance.

\ddagger This magnitude of the arc kB has been inferred from the following circumstances, communicated by Mr. MUDGE. 1st. "The whole

has arrived at the extremity of the semi-arc $OB = 135^\circ$, the auxiliary spring u will have been wound up through the same angle $= 135^\circ$, i. e. 27° , by the force of the main-spring acting on the pallet p , and 108° by the balance itself carrying along with it the arm GO , or Gk , while it describes the arc kB . The balance therefore returns through the arc BO by the joint action of the balance spring, and the auxiliary spring u ; the acceleration of both springs ceasing the instant the balance arrives at the quiescent point O ; when the balance has proceeded in its vibration about 27° beyond the point O to the position Ck , the rod ZW meets with the arm lk , and by carrying it forward, releases the tooth of the balance-wheel from the pallet q ; the balance-wheel accordingly revolves, and the upper tooth meeting with the pallet p , winds up the auxiliary spring u as before. The balance, with the crank, proceeding to describe the remaining part of the semi-arc kE , winds up the spring v through the further angle $kCE = 108^\circ$; the balance returns through the semi-arc EO , by the joint action of the balance-spring and the auxiliary spring v , both of which cease to accelerate the balance the instant it has arrived at O .

It is remarkable, according to this construction, that no force or impulse whatever is communicated to the balance from the main-spring, and yet the vibrations are continued of their due length: on further consideration it appears, that the maintaining power of the machine, instead of communicating any force

whole arc of vibration is about $\frac{1}{2}$ of a circle, or 270° , consequently the semi-arc of vibration is 135° . 2dly. The action of the balance, by carrying round the arm OG from k to B , winds up the auxiliary spring through an angle about four times greater than the angle OCk , through which it is wound up by the action of the balance-wheel on the pallet." Wherefore, if OB , or 135° , is divided into five parts, one of them will be $Ok = 27^\circ$, and the other four parts, or $kB = 108^\circ$.

or

or impulse, acts by removing a part of the force which retards the balance while it is describing the latter semi-arc of each vibration.

In the preceding account it has been shewn, that the balance describes the semi-arc from B to O by the joint action of the two springs; now for a moment let it be supposed that the balance vibrated through the entire arc BOE (fig. 85, pl. 5, and fig. 86, pl. 6.) by the joint action of the two springs, in the same manner as if one balance-spring only was applied of the same strength with both; in this case, the balance commencing its vibration at the extremity of the arc B, after having passed the semi-arc BO with an accelerated motion, would describe an equal arc OE on the other side of O, by retarded motion, provided it was not obstructed by friction, or other irregular resistances; but such resistances taking place will cause the latter semi-arc, which is described by retarded motion, to fall short of the arc OE by some small difference ES. There are two modes by which this latter semi-arc may be restored to its due length OE; either by communicating an impulse to the balance from the main-spring, or by removing a part of the force which retards the ascent of the balance while it describes the latter semi-arc OE. Mr. MUDGE has discovered and applied to his time-keeper the latter mode of supplying the power which is required to continue the balance in motion. During the progress of the balance through the semi-arc BO, it is accelerated by the joint action of the balance spring and auxiliary spring *u*; but while it describes the latter semi-arc OE (fig. 85 and 86), it is retarded by the joint actions of the balance spring and auxiliary spring, only while it describes a part of the semiarc from *k* to E, the retardation of the auxiliary spring being removed while it describes the first 27° of this semiarc from O to *k*.

It

It is evident, according to Mr. MUDGE's construction, that the diminution of retardation; which is equivalent to the supply of power to the balance, must always be of the same magnitude, so far as regards the influence of the main spring, provided it has sufficient force to wind up the auxiliary springs through the constant angle $OGk = Ock$ (fig. 86.); and the effects of heat and cold on the auxiliary springs are necessarily included in the compensation which is applied for heat and cold to the balance spring. The maintaining power, therefore, by which the motion of the balance is continued, must be always uniformly the same; this is an object usually held to be of material consequence in the construction of watches, and though often attempted by ingenious persons, has probably been accomplished in its full extent, for the first time, by Mr. MUDGE.

This construction possesses the further advantage of having the balance perfectly detached from the wheel work of the machine; the only communication* between the balance and the balance wheel is that which subsists while the pallet is disengaged from the tooth; an instant of time in a practical sense is almost evanescent: it should also be remarked, that the pressure

* The exact times in which the balance describes any portions of the arc of vibration may be readily obtained by having recourse to the theorem investigated in page 162, vol. 1. If we can ascertain the arc described by the balance while the tooth of the balance wheel is released from the projection of the pallet, the portion of time in which the balance is connected with the wheel-work of the machine will be known. By an experiment made on a very exact model of Mr. MUDGE's construction, it appeared that the balance described an arc of about 8° , while the tooth was released from the pallet. And since the balance describes 108° of its semi-vibration before the pallet begins to be moved, it will have described 116° before the tooth is released; we are therefore to find the times in which the balance describes severally the two arcs 108° and 116° ; the difference of these times will be the time in which the balance wheel is released from the pallet. By the theorem referred to,

Joint force of both springs to accelerate the circumference at the tension 90° ,
 or $f + \frac{1}{2}f = \dots \dots \dots 1.0040892$.

With * this force the balance vibrates five times in one second, when adjusted to mean time; the daily rate will therefore, in this case, be $= 0$.

(To be Continued.) ART.

* From the investigation in page 164, vol. 1. it appears, that if F is the accelerative force on the circumference of the balance at the angular distance from quiescence c° , $p = 3.14159$, &c. r = the radius of the balance expressed in inches, $l = 193$ inches; the time of a semi-vibration will be $= \sqrt{p^2 r c^\circ \div (8 l F \times 180^\circ)}$. In the present case $c^\circ = 90^\circ$, $r = 1$ inch, $F = f + \frac{1}{2}f = 1.0040892$; wherefore the time of a semi-vibration $= \sqrt{p^2 r c^\circ \div (8 l F \times 180^\circ)} = \frac{1}{5}$ part of a second precisely.

If the balance should vibrate by the force of the auxiliary springs only, the force of acceleration on the circumference at the distance 90° from quiescence is $\frac{1}{2}f = 0.0478138$; wherefore $\frac{1}{2}f$ being substituted for F in the quantity $\sqrt{p^2 r c^\circ \div (8 l F \times 180^\circ)}$, the other values remaining the same as in the former case, the time of a semi-vibration will now be $\sqrt{p^2 r c^\circ \div (8 l \times \frac{1}{2}f \times 180^\circ)} = 4.5825$; and the time shown by the watch in any portion of mean time t will $= t \div 4.5825 = t \times .21822$. Thus if t should be taken $=$ to one minute, or 60 seconds of time, the time shown by the watch in 60 seconds will be $= 60 \times .21822 = 13$ seconds nearly.

This would be the result if the forces of the balance and auxiliary springs were precisely in the proportion of 20 to 1. Mr. MURDOX mentioned this proportion by estimation only, not having any memorandum of experiments made to ascertain the exact proportion.

If, therefore, the actual rate of the watch should be observed when the balance vibrates by the action of the auxiliary springs only, it is probable that the time shown by the watch in 60 seconds of mean time might differ somewhat from that which has been here stated.

Since these notes were written, I have been favoured by his Excellency Count BRUHL with an account of an observation made on the rate of Mr. MURDOX's first time-keeper, when the balance vibrated by the action of the auxiliary springs only, the balance spring being removed. According to this observation, the watch showed

ARTICLE XXXIII.

Answers to the Mathematical Questions proposed in

ARTICLE XVIII. NO. VII.

I. QUESTION 131, answered by Mr. Burdon, Acaſter-Malbis.

LET x be the required number of days; then by the question,

$\frac{x+8}{15}$, $\frac{x-11}{21}$, $\frac{x-5}{27}$, and $\frac{x-18}{29}$ are whole numbers.

Suppoſe $\frac{x-8}{15} = p$, then $x = 15p + 8$. Therefore,

$\frac{x-11}{21} = \frac{15p-3}{21} = \frac{5p-1}{7}$ = a whole number; let it be taken from

$\frac{7p}{7}$, a whole number, and the remainder $\frac{2p+1}{7}$ is a whole

number: Multiply this by 4, then $\frac{8p+4}{7} = p + \frac{p+4}{7}$,

a whole number; and ſo is the remainder $\frac{p+4}{7}$ a whole num-

ber; let it be $= q$, then $p = 7q - 4$, and $x = 105q - 32$. This value being ſubſtituted for x in the third condition of the

question, we have $\frac{x-5}{27} = \frac{105q-37}{27} = 3q - 2 + \frac{8q-1}{9}$

= a whole number; and ſo is $\frac{8q-1}{9}$ a whole number; ſubtract

it from $\frac{9q}{9}$, and the remainder $\frac{q+1}{9}$ = a whole number = r ;

hence $q = 9r - 1$, and $x = 945r - 157$; then in the last

condition of the question, we have $\frac{x-18}{29} = \frac{945r-175}{29} =$

$32r - 6 + \frac{17r-1}{29}$ = a whole number; let the remainder be

multiplied by 2, and $\frac{34r-2}{29} = r + \frac{5r-2}{29}$ = a whole num-

ber; multiply the remainder by 6, then $\frac{30r-12}{29}$ a whole num-

ber, that is $r + \frac{r-12}{29}$, or $\frac{r-12}{29}$ a whole number = s ;

hence $r = 29s + 12$, and $x = 27405s + 11183$. If $s = 0$, then $x = 11183$ days, and consequently Mr. Milner is in the 31st year of his age.

The same by Mr. J. Collins, School-Master, Kensington.

Let the number which answers the two first conditions of the question be denoted by $15x + 8$, then will $\frac{15x+8-11}{21}$, or

$\frac{5x-1}{7}$ be a whole number, in which the least value of x is 3,

therefore $15x + 8 = 53$. Now the least common multiple of 15 and 21 being 105, it is evident that the number answering the three first conditions may be represented by $105y + 53$, then

$\frac{105y+53-5}{27}$, or $\frac{8y+16}{9}$ is a positive whole number, in

which the least value of y (being a whole number) is 7, hence $105y + 53 = 788$.

Lastly,

Lastly, the least common multiple of 15, 21, and 27 being 945, we may assume $945z + 788$ as the number sought, and

then $\frac{945z + 788 - 18}{29}$, or $\frac{17z + 770}{29}$ is a whole number, and

the least value of z is 11: wheref. $945z + 788 = 11183$ days, or $30\frac{2}{3}$ years, nearly.

The same, by Mr. Swale, Chester.

Let $x =$ Mr. Milner's age in days, then $\frac{x-8}{15}$, $\frac{x-11}{21}$,

$\frac{x-5}{27}$, and $\frac{x-18}{29}$ must be whole numbers; now put these

expressions equal to a , c , m , and n respectively, and then $x = 15a + 8 = 21c + 11 = 27m + 5 = 29n + 18$; from

equating the two first values of x , we get $a = \frac{21c + 3}{15} =$

$a + \frac{6c + 3}{15} = c + \frac{2c + 1}{5} = c + s$, putting $2c + 1 = 5s$;

but, since $c = \frac{5s-1}{2} = 2s + \frac{s-1}{2} = 2s + t$, where $2t =$

$s-1$, we have $a = 3s + t$, and $x = 45s + 15t + 8$. Again, by equating the third and fourth values of x , we get $m =$

$\frac{29n + 13}{27} = n + \frac{2n + 13}{27} = n + r$, by assuming $27r = 2n$

$+ 13$: but we have also $n = \frac{27r - 13}{2} = 13r - 6 + \frac{r-1}{2}$

$= 13r + v - 6$, where $v = r - 1$; hence $m = n + r = 14r + v - 6$, and consequently $x = 27m + 5 = 378r + 27v - 157$. Moreover, since $2t = s - 1$, and $2v = r - 1$, then

$s = 2t + 1$, and $r = 2v + 1$, and the two values of x , found above, will become $105t + 53$, and $783v + 221$, respectively.

Whence $105t + 53 = 783v + 221$, and $t = \frac{783v + 168}{105}$

$$= 7v + 1 + \frac{48v + 63}{105} = 7v + 1 + \frac{16v + 21}{35} = 7v + w$$

+ 1, by assuming $35w = 16v + 21$: but $v = \frac{35w - 21}{16}$

$$= 2w - 1 + \frac{3w - 5}{16} = 2w + d - 1, \text{ where } 16d = 3w -$$

$$5; \text{ now } w = \frac{16d + 5}{3} = 5d + 1 + \frac{d + 2}{3} = 5d + h + 1,$$

putting $3h = d + 2$, or $d = 3h - 2$. Conseq. $w = 16h - 9$, $v = 35h - 21$, $t = 261h - 155$, and $x = 105t + 53 = 27405h - 16222$, where h may be any integer whatever. If $h = 1$, $x = 11183$.

According to one or other of these methods the question was also answered by Messrs. Bosworth, Evans, Gregory, Hartley, Hill, Johnson, Lowry, M'Doneld, Marrat, Miss Susan May, Merones Minor, Milner, Reed, and Thornoby.

II. QUESTION 132, answered by Mr. William Francis, Teacher of the Mathematics at Hamstead, Middlesex.

Suppose the rectangle's breadth = 2, the circle's diameter will be = 10; also $2\sqrt{(12^2 - 2^2)} = 23.6642 =$ the rectangle's length. Hence the area of the rectangle = 47.3284 : but $10^2 \times .7854 = 78.54$ the circle's area; therefore the error is 31.2116 too little.

Again, suppose the breadth of the rectangle to be 4, then the circle's diameter will be 8, and the error will be 40.244 too much.

Whence, by the rule of position, 2.8736 will be found an approximate number for the breadth of the rectangle, and using this

this for another supposition, we find 2.834 for the breadth of the rectangle, very near; hence its length is 23.32, and the circle's diameter 9.166; also the area of each = 66 nearly.

The same answered by Merones Minor.

Fig. 324, Pl. 20. Let ABC be the given semicircle, DEFG the inscribed parallelogram, and BKHL the circle inscribed in the remaining segment. Put $a = 12 = \text{rad.} = \text{AI} = \text{BI}$, $p = .7854$, and $x = \text{IH} = \text{DE} = \text{FG}$; then $a - x = \text{BH}$, $\sqrt{a^2 - x^2} = \text{HE}$, and the area of the parallelogram is $2x\sqrt{a^2 - x^2} = (a - x)^2 \times p$, the area of the circle. Whence by *Em Alg. Rule 5, Pa.* 283, I find $x = 2.8311$, very near, and $\text{BH} = 9.1689$; also $\text{EF} = 2\sqrt{14.8311 \times 9.1689} = 23.3224$.

The same, by Mr. John Blackwell, Hungerford.

Put $a = 12$, the semidiameter of the semicircle, $b = .7854$, and $x =$ the diameter of the circle required, then will $a - x =$ the br. of the parallelogram and $2\sqrt{a^2 - (a - x)^2}$ its length; theref. $(a - x) \times 2\sqrt{a^2 - (a - x)^2} =$ the area of the parallelogram, and $bx^2 =$ the area of the circle. Hence $(a - x) \times 2\sqrt{a^2 - (a - x)^2} = bx^2$. This equation reduced, produces a cubic, wherein $x = 9.1689$. Whence the length of the parallelogram is easily found.

The same, by Mr. J. Hartley, Auditor's Office, Somerset Place, London.

Make the rad. of the semicircle $= 12 = a$, $.7854 = c$, $\text{BH} = x$, and $\text{EF} = y$, then will $a - x = \text{HI}$, and, per question, $\text{EF} \times \text{HI} =$ area of the circle BLHK, i. e., $cx^2 = ay = xy$;

also, by Eu. I. 47. $\sqrt{a^2 - (a - x)^2} = \frac{1}{2}y$, or $y = \sqrt{8ax - 4x^2}$. Whence $\sqrt{8ax - 4x^2} = y = cx^2 \div (a - x)$. Hence, from the resolution of a cubic equation, $x = 9.1689$, $a - x = 2.8311 = \text{HI}$, and $y = cx^2 \div (a - x) = 23.3224 = \text{EF}$.

This question was also answered by Messrs. Bosworth, Byerley, Collins, Evans, Gregory, Harris, Hill, Johnson, Lowry, Marrat, Miss May, Reed, Swale, and Hornoby.

III. QUESTION 133, answered by Miss Susan May, Birmingham.

Three numbers answering the conditions of the question may be found by the following simple process. Let a^2 be the given

square, and put $\frac{x}{2} + a$, $\frac{x}{2} - a$, and $2x$ for the numbers required.

The three first conditions are evidently fulfilled, $\left(\frac{x}{2} + a\right) \cdot \left(\frac{x}{2} - a\right)$

$+ a^2 = \left(\frac{x}{2}\right)^2$, $\left(\frac{x}{2} + a\right) \cdot 2x + a^2 = (x + a)^2$, and

$\left(\frac{x}{2} - a\right) \cdot 2x + a^2 = (x - a)^2$, being all squares. We have,

therefore, only to make the product of the three numbers when added to the given square, a square number, that is,

$\left(\frac{x}{2} + a\right) \cdot \left(\frac{x}{2} - a\right) \cdot 2x + a^2 = \frac{x^3}{2} - 2xa^2 + a^2$, must be a

square; assume its side $= xa - a$, then $\frac{x^3}{2} - 2xa^2 + a^2 = x^2a^2$

$- 2xa^2 + a^2$; hence $x = 2a$, and the three numbers are $a^2 + a$, $a^2 - a$, and $4a^2$.

If $a^2 = 2$, the numbers are 6, 2, and 16, and the four squares 4, 6, 10, and 14.

The same answered by Mr. George Reed, pupil to Mr. Thomas Bulmer; Sunderland.

Let x , y , and z be the three numbers sought, and a^2 the given square. Then by the question,

$$xy + a^2$$

$$\left. \begin{aligned} xy + a^2 &= \square = r^2, \text{ or } xy = r^2 - a^2 \\ xz + a^2 &= \square = n^2, \\ yz + a^2 &= \square = m^2, \\ xyz + a^2 &= \square = v^2, \text{ or } xy = \frac{v^2 - a^2}{z}; \end{aligned} \right\} \text{Hence } r^2 - a^2 = \frac{v^2 - a^2}{z}$$

or $z = \frac{v^2 - a^2}{r^2 - a^2}$, and, by substitution, $\frac{v^2 x - a^2 x}{r^2 - a^2} + a^2 = n^2$, or

$$x = \frac{r^2 n^2 - n^2 a^2 + a^4 - a^2 r^2}{v^2 - a^2}; \text{ and, again, by substitution,}$$

$$\frac{v^2 y - a^2 y}{r^2 - a^2} + a^2 = m^2, \text{ or } y = \frac{m^2 r^2 - m^2 a^2 + a^4 - a^2 r^2}{v^2 - a^2}.$$

And if $a = 2$, $r = 3$, $v = 8$, $n = 4$, and $m = 8$; then $x = 1$, $y = 5$, and $z = 12$, the three numbers required.

Thus the question was answered by Messrs. Bosworth, Evans, Gregory, Hill, Johnson, Lowry, Simpson the proposer, and Thornoby.

The same otherwise by Mr. I. H. Swale.

Let x , y , and z be the three required numbers, and m^2 the given square. Then must xy , xz , yz , and $xyz + m^2$ be transformed into squares. Now since xy and xz are squares, their product $x^2 yz$ will be a square, and consequently yz will be a square, in all cases wherein xy and xz are such. Hence, we have only to make \sqrt{xy} , \sqrt{xz} , and $\sqrt{m^2 + xyz}$, rational numbers, whole or

fractional. Assume $xy = r^2$, and $xz = v^2$; then, $x = \frac{r^2}{y} = \frac{v^2}{z}$,

$y = \frac{r^2 z}{v^2}$, and $xyz = zr^2$; also the formula $m^2 + xyz$ will become,

by

by substitution, $m^2 + zr^2$, a square, suppose it equal to $m^2 +$

$2ms + s^2$. Whence $z = \frac{s^2 + 2ms}{r^2}$, $y = \frac{r^2 z}{v^2} = \frac{s^2 + 2ms}{v^2}$,

and $x = \frac{r^2 v^2}{s^2 + 2ms}$, where r , s , v , and m may be taken at

pleasure.

If $r = 2$, $s = 3$, $v = 4$ and $m = 5$, we shall have $x = \frac{64}{39}$,

$y = \frac{39}{16}$, and $z = \frac{39}{4}$; also, $xy = 4$, $xz = 16$, $yz =$

$\frac{1521}{64}$, and $m^2 + xyz = 64$; all squares, the sides of which

are 2, 4, $\frac{39}{8}$, and 8, respectively.

And thus the question was also answered by Messrs. Collins, Hartley, M'Donell, and Merones Minor.

IV. QUESTION 134, answered by Mr. T. Barber, St. John's College, Cambridge.

Analysis. Fig. 325, Pl. 20. Let F be the given point, and BG the required chord. Join F with C the centre, and produce FC both ways to the points H and D, taking $DF : FH :: GF : FB$; join also HB and DG, then the Δ s HBE and GFD are similar. Also take $DA : HE :: DF : FH$ and join BE, AG, then since $DF : FH :: DA : HE$, $DF : FH :: AF : FE :: GF : FB$, theref. the Δ s AGF, BFE are similar, and the $\angle AGF = EBE$, theref. the whole $\angle HBE =$ the whole $\angle AGD$, that is to a right angle; Whence this construction.

Through the given point F draw the diameter DA which produce to H, taking $DF : FH$ in the given ratio, take also $DA : HE$ in the same ratio; on HE as a diameter describe a semicircle cutting

ting the proposed circle in the point B, join BF and produce it till it meets the circumference at G and BG will be the required chord.

The same answered by Mr. Louis Hill.

Let $m : n$ be the given ratio. Through the given point P (fig. 326, pl. 20.) draw the diameter APV, let O be the centre of the given circle. On PA produced take $DP : PO :: 2m : n - m$, and on DP as a diameter describe a semicircle to intersect the given circle in E, draw EPF and it is the line required.

Demon. Join DE, and upon EF demit the \perp OI; thereby sim. Δ s, $EP : PI :: DP : PO :: 2m : n - m$, by conf. therefore by composition, since $EI = IF$, $EP : PF :: m : n$, i.e. EP, PF are in the given ratio. *Q. E. D.*

Remark. Since the semicircle may cut the given circle in two points, there will be two lines answering the problem.

The same, by Mr. Johnson.

Through the given point P (fig. 326, pl. 20.) draw the diameter APV, and take AP to PB in the given ratio, from P apply PE such that its square may = the rect. VP·PB, produce EP to meet the circle in F, and EF is the chord required. For by construction,

$VP : EP :: EP : PB$, and by the circle $VP : EP :: FP : AP$; therefore $FP : AP :: EP : PB$, or $FP : EP :: AP : PB$, i.e. in the given ratio.

The same by Mérones Minor.

Analysis. Fig. 326, Pl. 20. Let P be the given point and $m : n$ to the given ratio, n being the greater, and let EF be drawn as required. Through P and the centre O let the diameter AV be drawn. Then, by *Em. Geo. Cor. 2. to Prop. xx. B. IV*, $AP \times PV = EP \times PF = (n \div m) \times EP^2$, and consq. $EP = \sqrt{(AP \times m \div n \times PV)}$: Hence this construction:

Find PG a 4th proportional to n , m , and PV, and upon AG let a semicircle be described; make $PH \perp$ to AG, and from P let $PE = PH$ be applied to the given circle, and draw EPF the chord required.

Answers equally ingenious were given by Messrs. Burdon, Bofworth, Byerly, Collins, Evans, Gregory, Harris, Lowry, M'Doneld, Mathesis, and Swale.

V. QUES.

V. QUESTION 135, answered by Mr. J. Johnson.

Let ACB (fig. 327, pl. 20.) be the Δ required, and suppose CE to be drawn from the vertex to the middle of the base AB. Put $a^2 = \frac{1}{2}$ the sum, $d^2 = \frac{1}{2}$ the diff. of the squares of the given sides, $b =$ the sum of the base and \perp , and $x =$ the base; then $ED = d^2 \div x$, $CD = b - x$, and $EC^2 = (d^2 \div x)^2 + (b - x)^2$; hence $AE^2 + EC^2 = x^2 \div 4 + (b - x)^2 + (d^2 \div x)^2 = a^2$, and by reduction we get $5x^4 - 8bx^3 + (4b^2 - 4a^2)x^2 + 4d^4 = 0$, from which equation x may be determined.

The same, by Merones Minor.

Put $s =$ the sum of the sides, $d =$ their diff. $x =$ the base, and $a - a = \perp$. Then by *Em. Geo.* 38. II. *Cor.* 2, $\frac{1}{2}(s^2 - x^2) \times \frac{1}{2}(x^2 - d^2) = (\text{area})^2 = (a - x)^2 \times \frac{1}{2}x^2$, consequently $5x^4 - 8ax^3 + (4a^2 - s^2 - d^2)x^2 + s^2d^2 = 0$. Whence x may be determined.

Ingenious answers were also given by Messrs. Bosworth, Gregory, Hartley, Hill, Lowry, Swale, and Thornoby.

VI. QUESTION 136, answered by Miss Susan May.

Let ACB (fig. 328, pl. 20.) be the Δ , then since the $\angle ACB$ is 36° , or $\frac{1}{10}$ th part of the circle, AB is the side of a decagon inscribed in the circle whose radius is AC or CB. Now by *Em. Geo. B.* iv. *Pro.* 47, *Cor.* 2, $AC : AB :: 2 : \sqrt{5} - 1$, hence, by *sim. Δ s*,

As $4 + \sqrt{5} - 1 : 2 :: 16 + 8\sqrt{5} : 4 + 4\sqrt{5} = AC = BC$
 $\quad \quad \quad =$ the sides,

and $4 + \sqrt{5} - 1 : \sqrt{5} - 1 :: 16 + 8\sqrt{5} : 8 = AB =$ the base.

The same, by Mr. Bosworth, Cambridge.

Let the required Δ be represented by ABD, Fig. 329, Pl. 20. The sides of this Δ may be determined, without referring to trigonometrical tables or any other known values of sines and tangents in the following manner. Since the Δ is an isosceles one, and the vertical $\angle = 36^\circ$, each \angle at the base $= (180^\circ - 36^\circ) \div 2 = 72^\circ$, which is double that at the vertex. On AB set off AC equal to BD, and draw CD; then, as appears from the demonstration of *Eu.* iv. 10. $BA \times BC = BD^2$. Let BC be denoted by x , and suppose $BD = 6$; then, $BA = 6 + x$, and $BA \times BC = 6x + x^2$,

$+x^2$, which by the above property $= BD^2 = 36$. This equation reduced gives $x = 3\sqrt{5} - 3$; whence AB or AD is easily found $= 3 + 3\sqrt{5}$, and the perimeter of this supposed Δ , which is similar to the given one $= 6 + 3 + 3\sqrt{5} + 3 + 3\sqrt{5} = 12 + 6\sqrt{5}$. But the perimeter of the given Δ is $16 + 8\sqrt{5}$, therefore by sim. Δ s,

$$12 + 6\sqrt{5} : \left\{ \begin{array}{l} 6 \\ 3 + 3\sqrt{5} \\ 3 + 3\sqrt{5} \end{array} \right\} :: 16 + 8\sqrt{5} : \left\{ \begin{array}{l} 8 = BD \\ 4 + 4\sqrt{5} = AB \\ 4 + 4\sqrt{5} = AD. \end{array} \right.$$

Mr. Gregory, the proposer, deduced a solution, nearly similar, from the same proposition in Euclid.

The same, answered by Mr. J. Hartley, London.

The vertical \angle being 36° (1-10th of a circle), it is plain, the base of the Δ is the side of a decagon, and each of its sides the radius of the circumscribing circle. By *Em. Geo. B. iv. Pr. 47. Cor. 2*, it is as rad. : side of the decagon :: $2 : \sqrt{5} - 1$. Make $x =$ rad. of the circle, $a =$ the given perimeter, then will $a - 2x =$ the base of the isosceles Δ . Therefore, by the above proportion, $x : a - 2x :: 2 : \sqrt{5} - 1$; hence $x = 2a \div (\sqrt{5} + 3) = (32 + 16\sqrt{5}) \div (3 + \sqrt{5}) = 12.94427 =$ one of the equal sides of the Δ , and $a - 2x = 8 =$ its base.

The same, by Mr. J. T. M'Doneld, Holland, near Wigan.

Let ABC (fig. 328, pl. 20.) be the isosceles Δ whose perimeter is $16 + 8\sqrt{5}$, and vertical 36° . Then to determine the sides, produce CA, CB, till CE, CF are each $=$ to half the perimeter, and through the points C, E, F let a circle be described, whose centre O, will be in CD \perp to AB. Bisect the arch FC in G, and draw the right lines OE, OF, EF, DF, DG, and CG. Now by *Eu. III. 20*, $\angle EOF = 2ECF = 72^\circ = 1.5$ th of the circumference, and arch DF $= 1.5$ th arch DC. Put the chord DF $= x$, and CF $= 8 + 4\sqrt{5} = a$, then CD $= \sqrt{(a^2 + x^2)}$, $2FH = 2ax \div \sqrt{(a^2 + x^2)} = EF = CG$, and DG $= \sqrt{[a^2 + x^2 - (4a^2x^2 \div (a^2 + x^2))]}$. Also by *Fm. Trig. B. i. Pr. 39. Cor. 2*, DG \times DF $= OD^2$, that is, $\sqrt{[a^2x^2 + x^4 - (4a^2x^4 \div (a^2 + x^2))]} = (a^2 + x^2) \div 4$: Whence $x = 5.505 =$ DF $=$ DI; FH $= 5.2855$; CH $= 16.115$; and CI $= 12.311$.
Hence

Hence, by sim. Δ s, $CH : CI :: EF : AB (= 8) :: CF \text{ or } CE : CB \text{ or } CA (= 4 + 4\sqrt{5})$.

The same, by Mr. W. Marrat, Teacher of the Mathematics at the Grammar School, Boston, Lincolnshire.

An isosceles ΔABC (fig. 328, pl. 20.) whose vertical \angle is 36° , is = to one of the Δ s of a decagon. Now the area of a decagon whose side is unity, is, by page 114, *Hutton's Mens.* $\cdot 76942088$, $\frac{1}{10}$ th of which, or $\cdot 76942088$ is the area of one Δ , which put = m . Also put $16 + 8\sqrt{5} = s$, and $AB = x$,

and $1^2 : m :: x^2 : mx^2 = \text{its area} = CI \times \frac{1}{2} AB = \frac{x}{2}$

$$4\sqrt{\frac{s-x}{2}} = \frac{x^2}{4}. \text{ Hence } x = 8 = AB, \text{ and } AC = BC$$

$= 12.944$.

The same otherwise by Mr. W. Burdon, Acafter Malbis.

The sine of 36° to rad. 1, is $\frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$, and that of 72°

is $\frac{1}{2}\sqrt{\frac{5+\sqrt{5}}{2}}$. And in every plane Δ it is, as the sum of

the sines of the three \angle s, is to its perimeter, so is the sine of any one of its \angle s, to its opposite side. This proportion gives each of the equal sides of the $\Delta = 4 + 4\sqrt{5}$, and the other side = 8.

Messrs. Blackwall, Byerley, Evans, Frances, Harris, Hill, Johnson, Lowry, Merones Minor, and Peacock, gave neat solutions to this question.

VII. QUESTION 137, answered by Mr. J. T. M'Donell.

Let x = the radius of the cone's base, $p = 3.1416$, and s = its convex surface, then will the altitude be $= \sqrt{s^2 \div p^2 x^2 - x^2}$; also by *Emerson's Fluxions*, pa. 319, the distance between the centres

centres of oscillation and suspension, will be $= (4s^2 - 3\rho^2 x^2) \div 5\sqrt{(\rho^2 s^2 x^2 - \rho^4 x^4)}$, which, that the cone may vibrate the quickest possible, must be a minimum; but $(4s^2 - 3\rho^2 x^2) \div \sqrt{(s^2 x^2 - \rho^2 x^4)}$ is also a min. This expression being fluxed and the fluxional equation reduced, we get $x^2 + (s^2 \div \rho^2) \cdot x^2 = 4s^2 \div$

$$3\rho^2. \text{ Whence } x = \sqrt{\frac{s^2}{\rho^2}} \sqrt{\frac{19}{12} - \frac{s^2}{2\rho^2}}.$$

When $s = 100$ square inches, then $x = 5.265$, the altitude $= 2.9723$, and the distance from the vertex to the centre of oscillation $= 4.24296$. Hence the centre of oscillation falls 1.27066 inches below the cone's base; and when the radius of the base is equal to the altitude, the centre of oscillation falls exactly in the base's centre.

It may not be improper to observe here, that, the investigations of those authors, who make the distance of the centre of oscillation, when suspended by its vertex, to be *constantly four-fifths* of the altitude, are erroneous; as they consider all parts of the cone's base to move with equal velocity, which is evidently not the case.

And thus the question was answered by the proposer, Mr. W. Painsman, Teacher of the Mathematics at Hull, who also favoured us with the following different solution.

If $y = OI$, the semibase of the cone, $x = AO$ its altitude, $c = 3.1416$. Then the distance from this centre of motion to

that of oscillation, by *Simpson's Fluxions*, pa. 225, is $= \frac{4x}{5c} +$

$\frac{y^2}{5x}$, which must be a minimum that the cone may vibrate the

swiftest possible, or in this case $4x^2 + y^2 \div x$ is a min. Then $2cy$ $=$ periphery of the cone's base, and $cy\sqrt{x^2 + y^2} = s$, the convex surface in square inches; and $y\sqrt{x^2 + y^2} = s \div c$, therefore

$$y^2 + xy^2 = s^2 \div c^2 = n^2, \text{ or } y^2 = -\frac{x^2}{2} + \sqrt{n^2 + \frac{x^4}{4}}.$$

now substituting for y^2 , in $4x + y^2 \div x$, we have $\frac{7x}{2} +$

$\sqrt{\frac{x}{2} + \frac{x^2}{4}}$, a min. This being put into fluxions and reduced

gives $x = \sqrt{-\frac{17n}{8} + \sqrt{\frac{n^2}{3} + \frac{17}{8}n^2}}$. If the convex

surface in square inches be 100, then $x = 2.97233$, $y = 5.26566$, slant side $= 6.046$, and the distance from the centre of oscillation from A $= 4.24$ nearly.

Solutum. The result of this solution militates against what has been asserted, on this subject, by some great Mathematicians: *Stone* in his fluxions, pa. 178, makes the centre of oscillation to be 4.5ths of the altitude of the cone; but this answer is evidently erroneous, for he supposes that every particle of matter in the base of the cone moves with the same velocity, which is clearly not the case. And even Mr. *Emerson* in his Mechanics, avers that the centre of oscillation of a cone is 4.5ths its axis nearly. Now it appears from what is done above that it can only be so in one particular case; and that the centre of oscillation cannot fall within the cone at all unless the altitude be greater than the semidiameter of the cone's base; and when the altitude and semibase are equal, the centre of the base is the centre of oscillation, but when the semidiameter of the base exceeds the altitude it always falls below the base.

Other answers were given by Messrs. Barber, Bosworth, Byerly, Gregory, Hill, Johnson, Lowry, Mérones Minor, and Thornoby.

VIII. QUESTION 138. answered by Mr. Gregory, Cambridge.

To find the angle of the mountain's elevation supposing it to be conical of a regular form, the following method may be used: Let AIIH (Fig. 331, Pl. 20.) represent the mountain, from the vertex of which measure the right lines AB, AC, and proceed to measure on the plane towards the same point of the compass any convenient distance, as suppose to E and F. Upon the line BF erect perpendiculars as BC, DE, (which may be done various ways with the chain only) and continue them until they meet CE in C and E. Then (FB and CE being supposed produced till they meet in G, in the base of the mountain, directly under the vertex) the $\triangle s^{\circ}$ CKE and GBC being similar, as EK ($= DE - BC$): KC ($= BD$)

(= BD) :: BC : BG. Whence, in the $\triangle AGB$, right angled at G, AB, and BG are known, and consequently the angle ABG, the elevation of the mountain may be determined by a well known proportion.

The same answered by Mr. Francis.

Fig. 331, Pl. 20. Let A represent an object on the summit of the mountain, and ABD, ACE, two right lines carried down the side of the mountain to its edge at B and C, and continued from thence along the plane to D and E. Set off BC and DE at right angles to BD, (which, I presume, every surveyor can do with the help of the chain only); measure the distances DE, BD, BC, and BA. Then as DE — BC : BD :: BC : the horizontal distance between B and the point immediately under the mountain's top, or object A. We have, therefore, two sides of a right angled \triangle to find the perpendicular height of the mountain, which being known the area of the mountain's base, which is to be delineated instead of its superficies in the map, is readily obtained.

Similar answers were also given by Messrs. Bosworth, Burdon, Byerley, Lowry, Merones Minor, and Peacock.

IX. QUESTION 139, *answered by the proposer, Mr. John Harris, Teacher of the Mathematics and Land Surveyor, Carmarthen.*

By Emerson's Dialling, Sect. I. Pr. 10. If the dial be set up in any place, in a situation quite parallel to its first situation, it will always shew the time of the day in its original place: but, before that can be done, the latitudes and difference of longitudes of the two places must be known; and these being given, new requisites must be found to set it up by. Therefore, let BPH (Fig. 330, Pl. 20.) be the meridian, DHC the horizon, and B the zenith of the place it was made for; NS the meridian, and NE SW the horizon of the place A, it is to be set up at, and let P be the pole of the world and join AB with a great circle. Then in the spherical $\triangle ABP$ there are given, BP, the colat. of the place B, and AP = that of the place A, and the $\angle APB$ = their difference of longitude, to find AB = $\angle NDO$ = the inclination of the two horizons, or the co-reclination of the dial plane, at the place A. Also, to find the $\angle BAP$, = the supplement of its declination there.

Now, to find the elevation of the substile, or hour line of 12 on the dial, above the horizon = HD, we have given in the rt. \triangle spherical $\triangle POH$, HP = the stile's height, and the $\angle P$ = the

the earth, therefore $\sqrt{4.99996 \div 39.2} = .35714142$ will be the time of one vibra. of the shorter pendulum at the earth's surface. Now, since the square of the time varies inversely as the force on the same pendulum, and the force inversely as the square of the distance from the centre of the earth, the time varies directly at the distance; theref. $.35714142 : .35715 :: 3979 : 3979.095591$ the distance of the top of the rock from the earth's centre, and theref. $.095591$ miles $= 168.24$ yards, is the height of the rock, $\frac{2}{3}$ of which, 112.16 yards is that of the light-house.

The distance of the privateer from the foot of the rock may be found by this proportion, $1'' : 7.1428$ (time of 20 vibra.) $:: 2142$ (velo. of sound per sec.) $: 8157.0776$ feet, or 2719.0255 yards, the distance of the privateer.

Upon this principle is the answer given by Messrs. Bosworth, Burdon, Byerley, Lowry, and Merones Minor.

The same by the Rev. L. Evans.

By *Emerson's Mechanics* the distance of the centre of oscillation of the pendulum from the point of suspension will be $4.971 \times$

$$\frac{2 \times 6''}{3 \times 4.971} = 4.999 \text{ or } 5 \text{ nearly. Then as } \sqrt{5} : \sqrt{39.2} :: 60 :$$

$168 = \text{vib. in a minute, and } 168 \times 60 \times 5 = 50400 \text{ vib. in 5 hours on the surface of the earth. And } 50400 - 50399 = 1 \text{ vib. lost, by the diff. of gravity at the top of the light-house, and surface of the earth. And since the number of vib. in a given time is reciprocally as the distance from the earth's centre, we have } 50400 : 3979 :: 50399 : 3979.078949, \text{ theref. } 3979.078949 - 3979 = .078949 \text{ or } 138.95024 \text{ yards the height of the rock and light-house, } \frac{2}{3} \text{ of which is } 55.580096 \text{ yards the height of the light-house. The distance of the privateer is easily found.}$

And thus the question is answered by Messrs. Blackwell, and Francis, the proposer.

XI. QUESTION 141, answered by Merones Minor.

CASE I. ANALYSIS: Fig. 333, Pl. 20.

Suppose it true. Through F let PFQ be drawn \perp to CD meeting CD and AB in P and Q, and join BF, CF.

Since

Since $AB + CD : AD :: AB : DF$, by altern. $AB + CD : AB :: AD : DF$,
 and by division $CD : AB :: AF : DF$; but $AF : DF :: FQ : FP$;
 theref. by equality $CD : AB :: FQ : FP$;
 wherefore the $\triangle BAF = \triangle CDF$;
 but the $\triangle BEF = \triangle CEF$;
 wheref. the space $ABEF = \text{space } DCEF$;
 therefore the trapezoid $ABCD$ is divided into two equal parts by
 the line EF drawn from the middle of BC . *Q. Q. V.*

CASE II. ANALYSIS. Fig. 333, Pl. 20.

Suppose it true. Join AE , DE .
 Since $AB + CD : BC :: AB : CE$, by altern. $AB + CD : AB :: BC : BE$,
 and by division $CD : AB :: BE : CE$; hence the $\triangle ABE = \triangle DCE$;
 but the $\triangle AEF = \triangle DEF$; theref. the space $ABEF = \text{space } DCEF$;
 wherefore the trapezoid $ABCD$ is divided into two equal parts by
 the line EF drawn from the middle of AD . *Q. Q. V.*

The same, by Mr. Harris, Carmarthen.

Produce BA till $AG = AB$, and join AE , DE , BF , CF , and
 FG .

When BC is bisected the $\triangle BFE = CFE$, by Eu. 38. I.
 theref. the $\triangle CFD = FAB = FAG$, and the $\angle CDF =$
 FAG , by parallels; consequently $CD \cdot DF = FA \cdot AG = FA \cdot AB$.
 Therefore $AB : CD :: DF : FA$, and by composition $AB + CD :$
 $AD (DF + FA) :: AB : DF$.

And when AD is bisected, the $\triangle AEF = FED$, but the trapez.
 $ABEF = \text{trapez. } CDEF$; theref. the $\triangle s$ ABE , DCE are equal,
 and, by hyp. the $\angle s$ at B and C are right; consequently $AB \cdot BE$
 $= DC \cdot CE$, theref. $AB : CD :: CE : BE$; theref. by compos.
 $AB + CD : BC (CE + BE) :: AB : CE$. *Q. E. D.*

The same, by Mr. Swale.

Produce DA , CB so meet in P . Join BF , CF , AE , DE , and
 demit upon DE the perpendiculars BG , CH :

1. Since $BE = EC$, by hyp. the $\triangle s$ CFE , BFE , are =,
 Eu. 38. I.
 consequently the $\triangle CFD = \triangle BAF$, and conseq $FD \cdot CH = AF \cdot BG$.
 Hence, $DF : FA :: BG : CH :: PB : PC :: AB : CD$;
 wherefore componendo &c. $AB + CD : AD :: AB : DF$.

2. Since $AF = FD$, by hyp.; the $\triangle s$ DEF , AEF are equal,
 consequently, the $\triangle ECD = \triangle EBA$,
 and conseq. $EC \cdot CD = EB \cdot BA$.

Hence

Hence, $AB : CD :: CE : EB$,

theref. by comp. $AB + CD : BC :: AB : CE$.

Ingenious demonstrations were also given by Messrs. Bosworth, Burdon, Elliott, Evans the proposer, Gregory, Hill, Johnson, Lowry, and M'Doneld.

XII. QUESTION 142, answered by Mr. Lowry, Birmingham.

This question may be solved by various methods, a few of which are here exhibited. There is, however no method of describing the Δ , upon the surface of a sphere, with the given data, that I am yet acquainted with, except in some particular cases. The proposer, no doubt, intended this question for an illustration of the use of *Prob. 17, Book I. of Constructions* of his spherical Geometry, but whoever will examine the *construction* there given will find insurmountable difficulties in the application of it. The lines necessary to be drawn must pass through a *solid* sphere, and even supposing it possible to draw them in that manner, yet still as great a difficulty would remain in drawing a tangent from an *imaginary* point to touch the sphere at the point of contact of the circles. If we suppose the Δ to be projected on a plane, the prob. may be reduced to *Prob. XII. of Lawson on Tangencies*, but the calculation from thence would be exceedingly troublesome. The simplest method of construction that has appeared to me is the following:—

Take AB (fig. 334, pl. 20.) = the sum of the given segments, AI = their difference, and AC = the given sum of the sides, to the radius OA : draw the indefinite tangent AD to touch the circle at A ; bisect AO in Q , AB in R , and AI in S , and draw OQD , ORE , and OSF to meet AD in D , E , and F ; take AG a 4th proportional to AD , AE , and AF : draw OG to cut the circle in T , then AT = half the diff. of the sides; hence $AT + AQ$ = the greater side, and $AQ - AT$ = the less side.

For, by spherics and construction, As the tangent AD of half the sum of the sides, is to the tangent AE of half the base, so is the tangent AF of half the diff. of the segments of the base, to the tangent AG of half the diff. of the sides; whence the whole is evident.

The same may be determined otherwise thus.

In fig. 335, pl. 20; take AD , DB = the given segments of the base to the radius OD , on OD demit the \perp s AE , BF ; take the arch AH = the supplement of the given sum of the sides, and divide it in G , by prob. 37, *Simpson's Select Exercises*, so the sines of

of GH, GA, may obtain the same ratio EO to OF; erect the \perp arch DC, and make AC = the complement of GH and BC = the comp. of AG, then is ACB the Δ required.

This construction will appear evident by considering that the cosines of the sides are as the cosines of the segments of the base, which is a well-known property of spherical triangles.

If the difference of the sides, instead of their sum, be given, the sides are determined the same way as above.

The same answered by Mr. William Burdon.

By theo. 1. art. 75, pa. 85, Crackelt's translation of Maudit's Trigonometry, it is, as the tan. of $\frac{1}{2}$ the base of any sph. Δ , is to the tan. of $\frac{1}{2}$ the sum of the other two sides, so is the tan. of $\frac{1}{2}$ the diff. or $\frac{1}{2}$ the sum of the segments of the base formed by the \perp , according as it falls within or without the Δ , whence the diff. of the sides, and consequently the sides themselves will become known.

The same, by Mr. Rd. Elliott, Liverpool.

Fig. 396, pl. 20. — Let AD, DB, be the given segments of the base, and from D erect the \perp DC. From A and B, by p. 20. b. 1, con. in Mr. Howard's *Treatise on Spherical Geometry*, draw AC, BC, to meet DC, and be equal to the given sum of the sides, and it is done.

The same, by Mr. J. T. M'Donald.

Draw the great circle AB (fig. 396, pl. 20.), making AD, BD, the given segments of the base; at D erect the \perp great circle DF; then,

Case 1. When the given sum of the sides is *less* than the semi-circumference of the sphere: about B, as a pole, with the given sum of the sides, describe a circle EG, and by prob. 19, pa. 137, of Mr. Howard's *Spherical Geometry*, describe the circle EA to touch EG, and have its centre C in DF; join AC, BC, and the thing is done.

Case 2. When the sum of the sides is *equal* to the semi-circumference; let E be the point on the sphere opposite to B; draw the great circle EA, and bisect it by the \perp great circle DC, which will meet DF in C, the vertex of the Δ . For $EC + CB = AC + CB$, the given sum of the sides.

Case 3. When the sum of the sides *exceeds* the semi-circumference. With the given base AB, and sum of the supplements of AC,

AC, BC, let the opposite Δ AcB be constructed as in *Case 1*, and produce the sides till they meet in the opposite side of the sphere in C, the vertex of the Δ required. For $\angle C = \angle c$, and AC, CB, are the supplements of Ac, cB.

Ingenious constructions were also given by Messrs. Bosworth, Gregory, Hill, Howard, the proposer, Johnson, Merones Minor, and Thornoby.

XIII. QUESTION 143, answered by Mr. John Blackwell.

Let ABCD (fig. 337, pl. 20.) represent the given can, the solidity of which is easily found = 525.337043 cubic inches, EFCD the part immersed in the water; to find its dimensions, we have AB = 6.5, CD = 8.7, BH = 11.5, and FD = 2; now BD =

$$\sqrt{(BH^2 + \frac{CD-AB^2}{2})} = 11.55249, \text{ also, by sim. } \Delta s, \text{ as BD:}$$

BH :: FD : FG = 1.99094, and BH : $\frac{1}{2}(CD - AB) = HD :: GF : GD = .1904348$, therefore $CD - 2GD = EF = 8.3191304$, and the content of the frustum EFCD = 113.248024814. Hence, as 1728 : 1000 :: 113.248024814 : 65.53705189 the weight of the can; and, as 7300 : 1728 :: 65.53705189 : 15.51842805 the solidity of the tin, which, taken from that of the whole can ABCD, leaves 509.82361495 for the content of the inside. Now put x = the thickness of the tin, then will the inside dimensions be $6.5 - 2x$, $8.7 - 2x$, and $11.5 - x$, and by Mensuration, $[(6.5 - 2x)^2 + (8.7 - 2x)^2 + (6.5 - 2x) \cdot (8.7 - 2x)] \times (11.5 - x) \times .2618 = 509.82361495$, this equation reduces to $x^2 - 19.1x + 101.94833x = 4.938066$, in which x is = .05, or 1-20th of an inch, nearly = the thickness of the tin.

The same, by the Rev. L. Evans.

By Mensuration, the content of the outer frustum is = 2006.635 \times .2618. And putting x for the thickness of the tin, $6.5 - 2x$, $8.7 - 2x$, and $11.5 - x$, are the dimensions of the inner frustum, and $(6.5 - 2x)^2 + (8.7 - 2x)^2 + 6.5 - 2x \times 8.7 - 2x \times 11.5 - x \times .2618$, or $(2006.635 - 1223.29x + 229.2x^2 - 12x^3) \times .2618$ is its content, and $2006.635 \times .2618 - (2006.635 - 1223.29x + 229.2x^2 - 12x^3) \times .2618 = (1223.29x - 229.2x^2 + 12x^3) \times .2618 =$ content of the tin; and $(1223.29x - 229.2x^2$

$229.2x^2 + 12x^3) \times .2618 \times 7300 =$ its weight, which, by hydrostatics, is equal to the weight of the water displaced. Now, $\sqrt{(BH^2 + DH^2)} = DB = 11.552$, and by sim. Δ s $DB : DH :: DF : DG = .1904$, and so $EF = 8.3192$: also $DB : BH :: DF : FG = 1.991$, and the content of the frustum $DFEC = 432.596516 \times .2618$, and its weight $= 432.596516 \times .2618 \times 1000$; therefore $(1223.29x - 229.2x^2 + 12x^3) \times .2618 \times 7300 = 432.596516 \times .2618 \times 1000$, and $x^3 - 19.1x^2 + 101.94x = 4.938$, which gives $x = .05$, or 1-20th of an inch nearly.

The same question was ingeniously answered by Messrs. Bosworth, Byerley, Gregory, Johnson, Marrat, the proposer, and Merones Minor.

XIV. QUESTION 144, answered by Mr. O. G. Gregory, Cambridge.

Find a mean proportional between AB and AC (fig. 338, pl. 20.) the length and breadth of the card, and let it be set from A to the opposite side CD , falling on the point a . On Aa let fall the perpendicular, BE , which, with the line Aa , will divide the card as required. For, let the piece ACa be removed to the situation BDb , and the piece AEB to the situation aeb ; and the square $BEeb$ is formed: Ee a side of which is evidently equal to Aa , which, by the construction, is the side of a square equal in area to $ABDC$.

Remark 1. When $CD = 2AC$, the point a would fall in the middle of CD , and divide the card into 3 triangular parts: and when CD is more than double AC , no line from B could be admitted perpendicularly upon Aa , without producing it; in which case the prob. becomes impossible.

Remark 2. In the case before us, as the parallelogram is already formed, the side of an equal square may be determined in a manner different from the common one of finding a mean proportional: thus, set off half the length of the parallelogram from A to G , and make $Ga = AG$, then is Aa the side of an equal square; as may thus be demonstrated.

By theorem 36, Geometry in Dr. Hutton's Course,
 $Aa^2 = AG^2 + Ga^2 + 2AG \cdot GC$, or, as $AG = Ga$, $Aa^2 = Ga^2$, and
 $Aa^2 = 2AG^2 + 2AG \cdot GC$: but $GC = AC - AG$, therefore
 $Aa^2 = 2AG^2 + 2AG \cdot (AC - AG)$, and consequently
 $Aa^2 = 2AG^2 + 2AG \cdot AC - 2AG^2 = 2AG \cdot AC$.

Whence, as $2AG = AB$, it is manifest that $Aa^2 = AB \cdot AC$.

Q. E. D.

Merones Minor answered this question.

XV. QUESTION 145, answered by Mr. O. G. Gregory,
the proposer.

From a little consideration of the nature of this question, and of the numbers proposed, it will be obvious that the number of chords is odd, and consequently, that 84, the greatest chord, is the diameter of the circle. Now, let 84, the diameter, be denoted by a , $168 + 84\sqrt{3}$, the sum of the chords, by b , and AB (see fig. 339, pl. 20.) by x : then (as may be easily deduced from prop. 37, pa. 73, *Emerson's Trigonometry*) $a : b :: x : (b \div a) x = BK$, whence, by Euc. 47. 1. $AB^2 + BK^2 = AK^2$, that is, $x^2 + (b \div a^2) x^2 = a^2$;

$$\text{from which we get } x = \sqrt{\frac{a^2}{1 + (b \div a^2)}} = \sqrt{\frac{7056}{8 + 4\sqrt{3}}} = 42$$

$\sqrt{2 - \sqrt{3}} = AB$. As 42 is the radius of the circle, and as $\frac{1}{2}$ radius $\times \sqrt{2 - \sqrt{3}}$ is known to be the sine of 15° , it is evident that the value of AB just found, is equal to twice the sine of 15° , or the chord of 30° : therefore the number of equal parts into which the circumference is divided, is 12, the number of chords 5, and their lengths, as may be soon determined, are 42, $42\sqrt{3}$, 84, $42\sqrt{3}$, and 42 respectively.

The same, by Mr. Merones Minor.

Draw the chord AB, diameter AK, and supplemental chord BK; put $x = AB$, then $AK = 84$, $BK = \sqrt{84^2 - x^2}$, and by *Em. Trig.* cor. 1. pr. 37. b. 11. $(AK \cdot BK) \div AB = (84 \div x) \sqrt{84^2 - x^2} = 168 + 84\sqrt{3}$, per quest. ; hence $x = 42\sqrt{2 - \sqrt{3}}$, the chord of 30° , and the number of chords was 5.

The same, by Mr. W. Marrat. Boston.

Since the circumf. is divided into an even number of equal parts, the greatest chord will manifestly be the diameter; therf. let $AK = 84 = d$, $168 + 84\sqrt{3} = a$, and $BK = x$, then $AB = \sqrt{(d^2 - x^2)}$, and by cor. 1. pr. 37, pa. 73, *Em. Trig.* $(AK \cdot BK) \div AB = \text{sum of all the chords, that is, } dx \div \sqrt{(d^2 - x^2)} = a$, which gives $x = \sqrt{a^2 d^2 \div (d^2 + a^2)} = 84 \cdot 192$, and $AB = \sqrt{84^2 - 762^2}$ also (by Cor. Eu. VI. 8.) $Am = AB \div BK = 5 \cdot 6383$, therf. $Bm = \sqrt{AB^2 - Am^2} = 25 = \frac{1}{2}$ the least chord. Now, by

by Trig. as $Bn = 42 : Bm = 21 :: \text{rad.} = 1 : \text{fin.} \angle AnB = 30^\circ$, and $180^\circ \div 30^\circ = 6$; conseq. the number of chords is 5.

The same by Mr. W. Burdon, Acafter-Malbis.

The number of equal parts the circumf. is divided into being even, the greater chord will be = the diam. of the circle. Now upon the rad. $AL = 42$ (see fig. 340, pl. 20.) describe a semi-circle, and join LD , LB , &c. then the arc AL will be cut into = parts in the points N , M , &c. also $BQ = LN$, $DP = LM$, &c. Hence, because the Δ s ANL , AML , &c. are mutually = and sim. to the Δ s LBQ , LDP , &c. respectively, the sum of LA , LM , LN , &c. is = the sum of LF , DP , BQ , &c. = $63 + 21\sqrt{3}$. Put x = the least chord LN , then $\sqrt{42^2 - x^2}$ = the greatest chord, and by the Theorem at pa. 8, in the Introduction to Dr. Hutton's Logarithms we have this proportion, $x : 42 :: x + 42 + \sqrt{42^2 - x^2} : 126 + 42\sqrt{3}$, which gives $42 + \sqrt{42^2 - x^2} = (2 + \sqrt{3}) \cdot x$, and $x = 21$ = sine of 30° , or $360^\circ \div 30^\circ = 12$, the number of = parts the whole circumf. is divided into. Whence, the number of chords is 5, and their lengths 42 , $42\sqrt{3}$, 84 , $42\sqrt{3}$, and 42 .

Neat solutions were given to this question by Messrs. Bosworth, Elliott, Evans, Lowry, M'Doneld, and Miss Susan May.

XVI. QUESTION 146, answered by the Proposer, Philo.

Let a = the vel. of the stream flowing through a given aperture,
 x = the vel. of the wheel after its motion is uniform,
 n = the number of particles impinging upon a single wheel board from the time of its entering to the time of its quitting the stream,
 and t = the time of immersion. Then it is evident that $(a-x) \cdot tn$ will be the force of the stream upon a single wheel board in any one revolution, and the number of wheel boards acted upon in a given time, is as the vel. (x), of the wheel; therf. $(a-x) \cdot tnx$ is as the whole force of the stream upon the wheel in a given time, which must be a *maximum*. But the time t is always inversely, and the number of particles n is always directly as the impingent velocity $a-x$, therf. $tn = 1$, and the above expression for the maxim. becomes $(a-x) \cdot x$, which gives $x = \frac{1}{2}a$, agreeable to Mr. Waring's Theory.

The same, answered by Mr. Lowry, Birmingham.

Put V = the velocity of the water, and v = the velocity of the wheel, then $V - v$ is = the relative velocity, or that with which the water strikes the wheel. Now, let the velocity be what it will, it is evident that, the quantity of water that strikes the wheel will always be as the absolute velocity of the water, that is, as V , and that the force of any single particle will always be as the veloc. with which it strikes, that is as the relative veloc. $V - v$, therf. the whole force or effect of the water upon the wheel is as $V \cdot (V - v)$; conseq. $Vv \cdot (V - v) = V^2v - Vv^2$, must, by the quest. be a *maximum*, or, since V is supposed to remain constant, $Vv - v^2$ must be a *max.*; therf. $Vv = 2vv$, and $v = \frac{1}{2}V$, agreeing with the Theory of Mr. Waring, and the accurate experiments of the ingenious Mr. Smeaton.

Other ingenious solutions were given by Messrs. Bosworth, Byerley, Gregory, Marrat, McDondel, and Merones Minor.

XVII. QUESTION 147, answered by Mr. W. Francis.

Let BF and CGE (fig. 341, pl. 20.) represent the two masts, and ABC the ray. Take the angle it forms with the surface of the water, which will be equal to the angle CBG, in which right angled triangle all the angles and the side BG will be then known. On the opposite side of the water, at D, in the direction of the two masts take the angles ADB, and ADC, from whence the angles CBD, and BCD will be easily obtained; the line BC being found by plane Trigonometry, BD and afterwards AD, or the water's breadth, will be obtained in a similar manner.

Mr. Gregory of Cambridge, after pointing out a mode of proceeding, nearly similar to the foregoing, gives an example as below:

Suppose the distance FE between the two masts of the vessel to be 6 yards, the angle CAD = $18^\circ 26' \frac{1}{4}$, CDA = $36^\circ 52' \frac{3}{4}$, and BDA = $24^\circ 26' \frac{1}{4}$. Then in the right angled triangle BGC, are given the angle B = CAD, and BG = FE, to find BC = 6.32455 , and CG = 2. In the triangle BCD, we have BC = 6.32455 , the angle CDB = CDA - BDA = $12^\circ 25' \frac{1}{2}$, CBD = CBG + GBD = CBG + BDA = $42^\circ 52' \frac{1}{4}$, and BCD = $180 - (CBD + BDC) = 124^\circ 41' \frac{1}{4}$, whence CD is found = 20. And lastly in the triangle ACD, all the angles and the side CD are known; therefore AD is readily determined, = 52 yards, breadth of the water required,

The

The same, by the Rev. L. Evans:

Let AD be the breadth of the water; CE and BF the masts. At A let the \angle of elevation CAD be taken; as also the \angle s BDF, CDE. Draw BG \parallel to EF. Then in the rt. \angle d \triangle BGG are given the side BG and all the \angle s to find the sides BC and GC. In the \triangle BCD are given the side BC and all the \angle s to find the side CD. In the rt. \angle d \triangle DEC are given the side CD and all the \angle s to find the sides DE and CE. And lastly, as $CG : BG :: CE : AE$; therefore $AE + ED = AD$ the breadth of the water required.

The same, by Mr. T. Barber, St. John's College, Cambridge.

Let AD the breadth of the water, EC and FB the masts; join CB which, by the question, will fall on the verge of the water at A, from B draw BG \perp to CE, join, also, CD, and DB. First let the observer take the comp. of the \angle CAD, from which the \angle CAD will be known, and therof. it's $=$, the \angle CBG, conseq. all the \angle s and the side BG in the \triangle BGC are known, BC may therof. be found. Secondly, let the complements of the \angle s CDA, and BDA be taken, whence the \angle s CDA, and CDB will be had, and also the \angle BCD, therof. all the \angle s and one side of the \triangle BCD are known, from which CB will be had and thence AD.

Ingenious solutions were given by Messrs. Bosworth, Bardon, Hill, Johnson, Lowry, Merones Minor, and Peacock.

XVIII. QUESTION 148, answered by Mr. John Lowry.

This prob. is a particular case of *Prop. D, Art. 34, Vol. I. of the Repository*, where it is shewn that the locus of the required point is a circle given in magnitude and position.

For, let $AC \times m^2$ (see fig. 342, pl. 20.) be equal to the given solid. Then by the question

$PA^2 \times AC + PB^2 \times BC + PC^2 \times AB = AC \times m^2$,
therefore $PA^2 + PB^2 \times \overline{BC \div AC} + PC^2 \times \overline{AB \div AC} = m^2$.
Let AB be divided in F,

so that $AF : FB :: BC : AC$, join CF and divide it in X,
so that $FX : XC :: AB : AC + CB$, join also AX, BX and PX.

Then by *Prop. X. Art 34, Vol. I. of the Repository*,

$PA^2 + PB \times \overline{BC \div AC} + PC^2 \times \overline{AB \div AC} = AX^2 +$
 $BX^2 \times \overline{BC \div AC} + CX^2 \times \overline{AB \div AC} + PX^2 \times (AC +$
 $BC + AB) \div AC = m^2$. But X is a given point, therefore
 AX, BX, CX are given; consequently PX^2 is given. Hence
2 A 2 with

with the centre X , and distance PX , describe a circle, so shall any point in its circumference answer the conditions of the question, which of course is indeterminate without some other *datum*.

The sum of the solids will be a *minimum* when P coincides with X , but the *maximum* is infinite.

Observations, similar in purport to the above, were made by Mr Gregory, and Mr. Swale, the proposer.

XIX. QUESTION 149, answered by Mr. W. Wallace, of Perth Academy.

Analysis. Fig. 343, Pl. 20. Let ABC be the Δ required, which, because the base, and vertical \angle are given, will be inscribed in a given circle. Let the line BD , which bisects the vertical \angle , meet the circle in F ; the arch FA will be equal to FC ; therefore, if the diameter FH be drawn, it will bisect the base AC in E ; wherefore DE is half the diff. of the segments AD , DC . Join HB meeting AC produced in K ; the Δ s FED , KBD are evidently equiangular, therof. $ED : DB = EF : BK$; now the ratio of twice ED to DB , and therof. the ratio of ED to DB , is by hypothesis given, and since the circumscribing circle and base are given, the line EF will also be given in magnitude, therof. BK is given in magnitude: and it is well known that, if AH be joined, $KH \cdot HB = AH^2$, that is, (since AH is evidently given) to a given space; therof. BH and KH are both given in magnitude. Hence the following construction:—

Upon the given base AC describe a segment of a circle AHC that may contain an \angle equal to the given vertical \angle : bisect AC at E , and draw the diameter HEF . Let the given ratio of the line bisecting the vertical \angle to the diff. of the segments of the base made thereby be that of $\beta\delta$ to twice $\delta\epsilon$: find bk a fourth proportional to $\delta\epsilon$, $\delta\beta$, and FE ; join HA , and in bA produced find h , so that $kh \cdot hb$ may be $=$ to AH^2 : place the chord HB in the circle equal to hb ; join AB , CB , and ABC shall be the Δ required.

Demon. Join FB meeting AC in D . By a well known prop. in Geom. $KH \cdot HB = AH^2 =$ (by constr. $kh \cdot hb$, also, by constr. $BH = bh$, therof. $KH = kh$, and $BK = bk$: Now the line BD evidently bisects the vertical $\angle ABC$, and the Δ s KBD , FED are equiangular, therof. DB is to DE as KB , or kb to EF , that is, by constr. as $\delta\beta$ to $\delta\epsilon$: whereof. $DB : DC = DA = \delta\beta : 2\delta\epsilon$, that is in the given ratio as required.

The same, by Mr. John Lowry,

Constr. Upon the given base AB (fig. 344. pl. 20.) let a segment of a circle be described capable of containing the given vert. \angle , let the base be bisected in F, and through F draw the diameter DFH. Through H let HI be drawn \parallel to AB, from F apply FI, such, that its ratio to 2FH may be $=$ to the given ratio of the line bisecting the vert. \angle to the diff. of the seg. of the base made by the said bisecting line. On FI, as a diameter, let a circle be described, and through its centre O let DEOK be drawn meeting the circumf. in E and K; from D, to the circle ABCD, apply DC = DE, and join AC, CB, so shall ACB be the Δ required.

Demon. Let DC, AB be produced to meet in L.

The Δ s HDC, FDL are similar, therf. DH : DC :: DL : DF, and by the circle DH:DK::DE:DF; therf. DC·DL = DK·DE; but DC = DE, by constr. therf. DL = DK, conseq. CL = EK = IF.

Hence, by sim. Δ s, As CG, the line bisecting the vert. \angle , is to FG, $\frac{1}{2}$ the diff. of the seg. of the base m^d by the said bisecting line, so is CL, or IF, to FH, that is, CG is to FG in the given ratio. Also, the base and vert. \angle s are $=$ the given ones by constr.

Q. E. D.

The same, by Mr. J. T. M'Doneld.

Analysis: Suppose the thing done, viz. that ACB (fig. 345 pl. 20.) is the required Δ , about which a circle being described bisect the base AB by the diam. DF, and draw CD the line bisecting the vertical \angle . Join CF and let it meet AB produced in H. Now the ratio CE to AE — BE being given, that of CE to EG ($\frac{1}{2}$ the diff. of the seg.) is likewise given, which is the ratio DG to CH, because the Δ s DGE, HCE are sim.: but DG, and therf. CH is given. Also, FH·HC = BH·HA, i. e. CH² + HCF = 2HAG + AH², and, by Eu. 47. 1. HF² = GH² + GF², i. e. CH² + CF² + 2HCF = AG² + AH² + 2HAG + GF²; hence CF² + HC·CF = AG² + FG², i. e. HF·FC = BF². Whence this construction:—

Upon the given base AB, describe a seg. of a circle to contain the given vert. \angle ; complete the circle, and bisect the base with the diam. DF, and join BF; then take GO to GD in the given ratio of the line bisecting the vert. \angle to the diff. of the segments of the base made thereby, and in GB produced take GK = BF. With the centre O, and rad. OG describe a circle, and from K draw the straight line KNL, to pass through the centre O, cut the

circle in N and to meet it in L. Lastly apply the distance KN from F to C which will determine the vertex of the Δ .

The same, by Mr. Swale, Chester.

Analysis. Let ACB (see fig. 257, pl. 17.) be the required Δ , the base AB being bisected in L by the diameter of its circumscribing circle EF, which is given in magnitude; draw CE, the line bisecting the vert. \angle , meeting AB. Demit upon AB, AC, EF, EC, the \perp s CD, FH, CN, LQ. Then $CK:KL = CD:LQ$, and $EF:EL = FC:LQ$ are given ratios, for, EF and EL are given lines; hence $CD:DF$ is a given ratio, theref. $CD^2:CF^2$, i. e. $LN^2:(CF^2 =) FE \cdot FN$ is given, hence the prob. is reduced to this, viz. To cut the right line LF in N so that the square of LN may be to $EF \cdot FN$ in a given ratio, which is done at Prob. 3, of *Wales' Determinate Section*.

The same, by Merones Minor.

Analysis. Suppose ABC (fig. 346, pl. 20.) to be the Δ required, BD the line bisecting the vertical \angle , and AE the diff. of the seg. of the base made thereby; join BE, and through any point F, in the bisecting line BD, draw IFGH \parallel to AC. Then, by Em. Geo. 13, II. $BD:EA :: BF:GH$: but the ratio of BF to GH is given, therefore GH is given with respect to BF. Hence this construction:—

On any assumed line BF, describe the rhombus BLFM, having the \angle LBM of the given magnitude; then by Simp. Geo. Prob. 21, draw LN, so that $NO = GH$, (BF being to GH in the given ratio) and through F draw IH \parallel to LN. In IH take IK = the given base, and draw KA \parallel to BI, meeting BH in A, and through A draw AC \parallel to IH, meeting BI in C, and ABC will be the Δ required.

Nearly similar to this was the construction given by Mr. Gregory.

The same, by Mr. Johnson, Birmingham.

Constr. Fig. 347, Pl. 20. Draw the lines AC, CB to include and CD to bisect the given \angle , and from any point E in CD draw EF, EI \parallel to AC, BC respectively; join FI and take IT to CE in the given ratio of the diff. of the seg. to the bisecting line; draw TS \parallel to CB, and make $IH \cdot HF = IS^2$. Now from H, to EI, apply $HG = IS$, join FG and let it meet AC in K, and in ML, draw \parallel to EK through E, take $MV =$ the given base; through V draw VB \parallel to AC and let it meet BC in B: through B draw AB \parallel to ML, and ACB is the Δ required.

Demon.

Demon. Join HK. By constr. $IH \cdot HF = IS^2 = HG$ theref. the Δ s GHI, HFG are sim. and so are the Δ s GHI, FKI hence the \angle GKI = GHI, and the points G, H, K, are in a circle, theref. the \angle GHK = GIK = TSI, and \angle KGH = KIH = TIS, wheref. the Δ s KGH, IST are equiangular, also $HG = IS$, theref. $KG = IT$. Now, by parallels, $KF = ME$ and $GF = EL$, therefore $ME - EL = KG - IT$; conseq. $CE : ME - EL$ is in the given ratio, and because the Δ s MCL, ACB, are sim. it follows that $CD : AD - DB$ is in the same given ratio.

Q. E. D.

And nearly in the same manner was the solution given by Messrs. Hill, Peacock, and Thornoby.

XX. QUESTION 150, answered by Mr. W. Wallace, Perth Academy.

In this question, by the rectangle under the sides and the line bisecting the base, the proposer probably meant the rectangle under the sum of the sides and the line drawn from the vertex to the middle of the base; this being supposed, the *Analysis* may be as follows. Suppose ABC (Fig. 348, Pl. 21.) the Δ required, let AC its base, which is given in magnitude, be also given in position, then because the vertical angle ABC is also given, the point B is in the circumference of a given circle: Let EF, the diameter, bisect the base AC in D, the point F will be given, and if FA be joined, the line FA will be given in magnitude. Join EB, DB; By a well known prop. in Geom. $AC : AF = AB + BC : BF = (AB + BC) BD : BF \cdot BD$, now AC, AF are given in magnitude, and, by hypothesis, $(AB + BC) \cdot BD$ is given, therefore, $EB \cdot ED$ is given. Let G be the centre of the circle; in EF produced take H so that $GD \cdot GH = GF^2$, bisect HD in K, and draw BL perpend. to EF; the points H, K will evidently both be given. Now by Prop 6, *Steward's General Theorems* (see Vol. I. page 28, of the Repository.).

$2DG : DB = DB : KL$, and from the nature of the circle, $EF \cdot FB = FB \cdot FL$; theref. $2DG \cdot EF \cdot DB \cdot BF = DB \cdot BF \cdot KL \cdot FL$: Now the rect. $2DG \cdot EF$ is given, for DG and EF are given, and $DB \cdot BF$ has been shewn to be given, theref. the rect. $KL \cdot FL$ is given, and, since the points K, F are given, the point L is also given. Hence the following construction:—

Let AC be the given base and P the rectangle contained by the sum of the sides and the line drawn from the vertex to the middle of the base: Upon AC describe a segment of a circle that may contain an \angle equal to the given vertical \angle ; draw the diameter EF bisecting AC at D, join AF. Let G be the centre of the circle,

circle, and, GF produced, find H so that $HG \cdot GD = GF^2$ (or, which is the same, draw AH touching the circle) and bisect DH in K. Find a line Q so that $AC : AF = P^2 : Q^2$; and in K, produced, find the point L so that $2DG \cdot EF : Q^2 = Q^2 : KL \cdot FL$; draw LB perpend. to EF, meeting the circle in B; join AB, BC, and ABC shall be the Δ required.

The construction would have been nearly the same, had the difference of the sides been taken instead of the sum.

The same, by Mr. John Lowry.

Analysis. Suppose it done, and that ACB (fig. 349, pl. 20.) is really the Δ required. Let O be the centre of the circumscribing circle, and FODG a diameter \perp to, and bisecting the base in D. Join OA, GB, and drawn CE \parallel to AB, and EV \parallel to GB, draw also ES to make the $\angle ESD = \angle AOD = \angle ACB$, and AX to make the $\angle AXD = \angle EVD$. Let $AB \times m$ be $=$ to the given rect. under the sum of the sides and the line drawn from the vertical \angle to the middle of the base. Then, $AC + CB : AB :: m : DC$, and by *Pr. 12 and 18, Simp. Trig.* and *sim. Δ s*, $AC + CB : 2AB :: BV : AC + CB :: m : 2DC$, and $IS : CD :: CD : AB$, (AD being bisected in I); therefore, $BV \times IS : (AC + CB) \cdot CD = AB \cdot m :: m : 2AB$, hence, $2BV \times IS = m^2$.

Again, because AD, and the \angle s AOD, AXD ($= \frac{1}{2} \angle ACB$) are given, OD and DX are given, therf. by *sim. Δ s* the ratio of DV to DS is given; let this be as DB : BH, (BH being drawn \perp to AB). On BH produced take BK = DI and KL = DS; then it will be as

BD : DV :: BH : KL, and by composition,

BV : BD :: BH + KL : BH; therefore,

$2BV \times IS (= m^2) : BD :: 2 \times (BH + KL) \cdot BL : BH$, or
 $2 \times (BH + KL) \cdot BL = (BH \div BD) \cdot m^2$. Hence this construction:—

Upon the given base AB describe the segment of a circle to contain the given vertical \angle , and draw the diam. FOD \perp to, and bisecting the base. Join A with the centre O, and draw AX to make the $\angle AXD = \frac{1}{2}$ the given vertical \angle . Let BL be made \perp to AB, in which take BH a 4th proportional to DX, DO, and DB, also let BK be taken $= \frac{1}{2}$ AD. By prob. 2nd, Wales's Determinate Section, find the point L so that $(BH + KL) \cdot BL = (BH \div 2BD) \cdot m^2$. Let LV be drawn \parallel to DH, draw also

The

VE to make the $EVD = \frac{1}{2}$ the given vertical \angle and let it meet DF in E, then if EC be drawn \parallel to AB, meeting the circle in C, and CA, CB be joined, ACB will be the Δ required.

The synthetic demonstration is evident from the preceding Analysis.

Note. The given rectangle must not exceed $2AF \cdot DE$, nor be less than $AB \cdot BD$.

In the same manner may the Δ be constructed when the rect. under the diff. of the sides and the line drawn from the vertical to bisect the base is given.

The same, by Mr. W. Peacock, the proposer.

Constr. Let a segment of a circle be described upon the given base AB, (fig. 350, pl. 21.) capable of the given vert. \angle , let AB be bisected in G, and through G let the diam. DF be drawn. Also let AF be joined, and upon DF let a rect. DSVF be constructed such, that its ratio to the given rect. may be the same as the ratio of DF to $2AF$. Now make $GQ = DG$, and produce FD to H, so that $QF \cdot DH$ may be $=$ to the square of DG, and draw DK \parallel to the line joining the points QV. By Simp. Geom. 18. V, make $HE \cdot DE = KF \cdot FV$, and draw EC \parallel to AB, meeting the circle in C, the vertex of the Δ required.

Demon. We have only to prove that $(AC + CB) \cdot CG$ is $=$ to the given rectangle, for AB is $=$ the given base, and ACB is $=$ the given vertical \angle by construction. On AC denote the \perp DP, then by construction,

$DE : FV :: KF : HE$, therefore,

$DE \cdot DF : FV \cdot DF :: KF \cdot FQ : HE \cdot FQ$; but by constr.

$HE \cdot FQ = DG^2 + DE \cdot FQ = CG^2$ (by a known prop.), and

$DC^2 = DF \cdot DE$; therefore, by the above proportion,

$DC^2 : FV \cdot DF :: KF \cdot FQ : CG^2$; but, by parallels,

$KF : DF :: VF : FQ$, theref. $FV \cdot DF = KF \cdot FQ$; hence

$DC^2 : FV \cdot DF :: FV \cdot DF : CG^2$, or

$DC : FV :: DF : CG$, theref. $DC \cdot CG = DF \cdot FV$; but, by sim. Δ s,

$AC + CB (= 2CP) : DC :: 2AF : DF$, therefore

$(AC + CB) \cdot CG : DC \cdot CG (= DF \cdot FV) :: 2AF : DF$; but, by constr.,

$DF \cdot FV$: the given rect. $:: DF : 2AF$; therefore by equality

$(AC + CB) \cdot CG$ is $=$ to the given rectangle. *Q. E. D.*

Mr. I. H. Swale, also answered it.

XXI. QUESTION 151, answered by Mr. Samuel Thornoby.

By a method similar to some of those given in answer to the Prize Question in Ladies' Diary 1798, let a $\triangle DCE$ (fig. 366, pl. 21) be determined such, that DC may be $=$ to the line bisecting the base, $CE =$ the line bisecting the vertical \angle , and DE multiplied by the square of the \perp CF a *maximum*. Produce CE to meet DG , drawn \perp to DB , in G , and draw $CI \perp$ to CG meeting GD produced in I ; through the points G, C, I , let a circle be described meeting DE , produced both ways, in A and B , and join AC, BC , so shall ACB be the \triangle required, as is evident, any demonstration being unnecessary in this place.

XXII. QUESTION 152, answered by Mr. W. Wallace, Perth Academy.

Analysis. Fig. 351, Pl. 21. Suppose the lines PD, PE drawn as required, viz. so that the $\angle DPE$ may be of a given magnitude, and the ratio of BD to CE also given. Join PB , and make the angle BPG equal to the given angle DPE , also PG equal to PB ; thus G will be a given point. Through the points P, B, G describe a circle meeting AB in H ; this circle will evidently be given by position; therefore the point H , and the straight line which joins the points G and H will be given by position; let GH meet PE in K . Because the angle BPG is, by construction, equal to DPE , therefore DPB is equal to EPG , or KPG , now PBH is equal to PGH , that is PBD is equal to PGK ; also by constr. BP is equal to PG , therefore the \triangle s PBD, PGK are equal to one another, and BD is equal to GK ; now the ratio of BD to CE is given, by hypothesis; therefore the ratio of GK to CE is also given; and since the points G, C , and the lines GK, CE are given by position, also the point P , from which is drawn the line PKE : therefore the problem is reduced to the *Sectio Rationis* of the Ancient Geometricians, the method of constructing which is well known.

Construction. Let the given ratio which BD is to have to CE be the ratio of α to β . Join P and B , either of the given points in the lines given by position. Draw PG so that the angle BPG may be equal to the given angle, and PG equal to PB , through the points P, B, G , describe a circle meeting AB in H , and join GH ; and from P draw PKE to cut HG, AC , in K and E so that GK may be to CE as α to β (this may be done as in *Simpson's Geometry*, 2d Edit. Prob. 37.), draw PD meeting AB in D so that the angle EPD may be equal to the given angle, that is to GPB ,

'GPB, and also BPD equal to GPK: the lines PD, PE shall be drawn as required.

For the angles PGK, PBD are equal, and since the angle GPK is equal to BPD, also the side PG to the side PB the Δ s PGK, PBD are equal to one another, and GK is equal to BD: Now by constr. GK is to EC as α to β , therefore also BD is to CE as α to β .

The same, by Mr. Lowry.

Constr. Join P and B, one of the given points (see fig 365, pl. 21.), and PF to make the \angle BPF = the given one, and meet AC in F. On PF (produced if necessary) take PI: BP in the given ratio of CE to BD; draw FG to make the \angle PFG = PBD, and draw also PQ parallel to FG. By Prob. V. B. I. *Determinate Section, by Wales*, Cut CQ in E so that, PQ:EF:EQ:CE::PF:PI, join EP, and draw PD to make the \angle BPD = EPF, and it is done.

By constr. the angle DPE = BPF = the given angle, also by constr.

PQ:EF:EQ:CE::PF:PI. But by sim. Δ s, EQ:PQ::FF:FG, therefore PQ:EF = EQ:FG;

hence EQ:FG:EC:EQ::FG:EC::PF:PI, or FG:PF::EC:PI. Again, by sim. Δ s, BD:BP::FG:PF;

Therefore, BD:CE::BP:PI, that is in the given ratio by constr.

Q. E. D.

XXIII. QUESTION 153, answered by Mr. John Lowry.

Analysis. Suppose ACB (fig. 352, pl. 21.) the Δ required. Let GH be a diam. of the circumscribing, \perp to, and bisecting the base at N, join CG bisecting the vertical \angle , and on it demit the \perp s AF, BE, also, from C, upon AB and GH demit the \perp s CD, CI: draw DK \perp to EF, and join ED, FD and HC, then by the question the Δ EDF must be a *maximum*. Now, because of the right \angle s ADC, AFC, the points A, F, D, C, are in a circle, therefore \angle ADF = ACF = $\frac{1}{2}$ \angle ACB, and, because of the right \angle s CDB, CEB, the points B, D, E, C are in a circle, therefore \angle EDK = ECB = $\frac{1}{2}$ \angle ACB, and therefore the \angle s ACB, FDE, having the \angle s at R equal, and the vertical \angle s also equal, are similar. Therefore Δ ABC : Δ EDF :: CD² : DK², but CD² : DK² :: GH² : CH² (= GH · HI) :: GH : HI, therefore Δ ABC : Δ : : GH : HI, and therefore HI \times Δ ABC = GH \times Δ EDF; but GH is given and constant, therefore when the Δ EDF is a max. the Δ ACB \times HI must be a max. i.e. AN · NI · IH

must

must be a max. Now it is evident, from *Eu. II. 5*, that $NI \cdot IH$ will be the greatest when NH is bisected at I , therefore $NI^2 \cdot AN$ must be a max. or its quadruple $AN \cdot HN^2$ a max. and consequently $HN^4 \cdot AN^2$, or its equal $HN^5 \cdot GN$ must be a max.

Hence, by the same method that *Simpfon* deduces his 17th Theorem on the Max. et Min. from the 16th, it will appear that $NH : GN :: 5 : 1$, when $HN^5 \cdot GN$ is a maximum. *Ergo Solutum.*

Divide the diam. GH at N so that $NH : GN :: 5 : 1$, bisect NH at I , and draw ANB and IC , both \perp to GH , meeting the circle in A, B , and C , join AC, BC , and ACB shall be the Δ required, as is evident from the preceding analysis.

The same answered by Mr. J. Fletcher, Hollinwood, near Manchester.

Let $AHCG$ be the given circle, ABC the Δ required, and let the figure (352, pl. 21.) be completed agreeable to the problem; then, since it is well known that the trapeziums $ACBG, FDEN$, are similar, and also their parts ABC, FDE , it follows that the $\Delta ABC : \Delta FDE ::$ the trap. $ACBG : \text{the trap. } FDEN :: GB^2 : NE^2 :: HG \times GN : HI \times GN :: HG : HI$; but the area of ABC is $=$ to $AN \times IN$, therefore that of FDE will be a 4th proportional to HG, HI , and $AN \times NI$, and as this is to be a maximum it will be so when $HI \times NI \times AN$ is a maximum, because HG is constant; and this it is evident can only be a maximum, when HI is $=$ IN ; in this case, therefore, the expression becomes $\frac{1}{2} HN^2 \times AN$, and when this is a maximum, it is evident that, $HN^2 \times AN$, or $HN^4 \times AN^2 = HN^4 \times HN \times NG = HN^5 \times NG$ must be a maximum, and this is known to be the case when NH is $= \frac{1}{2} HG$, and since it has been shewn that NI must be $=$ HI , a method of construction is manifest.

XXIV. QUESTION 154, answered by the proposer, Mr. W. Wallace, Perth Academy.

Let ABC (fig. 359, pl. 21.) be the Δ formed by lines joining the centres of the given circles, and let G be the given point through which one side of the Δ required is to pass. Find a point P , such, that PA, PB, PC being joined, these lines may be to each other as the radii of the circles to the centres of which the arc drawn (this may be done as in *Simpson's Geom. 2nd Edit. Prob. 31*). Join PG , and upon PG describe a segment of a circle that may contain an angle equal to PBC , let this circle meet the circle

circle whose centre is B in E, join EG meeting the circle whose centre is C in F. Join PE, PF, and draw PD meeting the circle whose centre is A in D so that the angle EPD may be equal to BPA, and so that EPB may be equal to DPA, and let the points D, E, F be either all upon the convex circumferences of the given circles or else all upon the concave circumferences; Join DE, DF, EF and DEF shall be the Δ required.

Draw the radii BE, AD. Because the angle BPE is, by constr. equal to APD, also $BP : BE = AP : AD$, the Δ s BPE, APD are equiangular, therefore $BP : PA = EP : PD$, and since the angles BPA, EPD are equal, the Δ s BPA, EPD are equiangular. The Δ s BPC, EPF are also equiangular, for the \angle PEF is by constr. equal to \angle PBC, and if \angle EPF be not equal to \angle BPC, suppose it to be either greater or less, and make the angle EPH equal to BPC, and join CH, then the angle BPE is equal to CPH, and, since by const. $BP : BE = CP : CH$, the Δ s BPE, CPH are equiangular, therefore $BP : PC = EP : PH$, wherefore the Δ s BPC, EPH are equiangular and the angle PEH is equal to PBC, that is, by constr. to PEF which is impossible, therefore EPF is not unequal to BPC, that is, it is equal to BPC, and the Δ s EPF, BPC are equiangular, therefore the remaining Δ s FPD, CPA are equiangular, and the whole Δ s EDF, BAC are equiangular.

XXV. QUESTION 155, answered by the proposer, Mr. James North.

The fluent of the exponential $xe^{\frac{2x}{a}}$ is easily found $= \frac{1}{2} ae^{\frac{2x}{a}}$.

Put y the sine, v the cosine, t the tangent, and s the secant of the arc z . Then $y\dot{y} = -v\dot{v} = vyz$, and hence the given fluxion

$$\frac{-\dot{z}}{v^3} = \frac{-\dot{y}}{v^4}. \quad \text{Again, putting } w = \frac{y}{v^2}, \quad \text{we have } \dot{w} =$$

$$\frac{v^2\dot{y} - 2vy\dot{y}}{v^4} = \frac{\dot{v}^2 + 2y^2\dot{y}}{v^4} = \frac{2v^2 + 2y^2 - v^2}{v^4}\dot{y} = \frac{2 - v^2}{v^4}\dot{y}$$

$$= \frac{2\dot{y}}{v^4} - \frac{\dot{y}}{v^2}. \quad \text{Hence } \frac{\dot{y}}{v^4} = \frac{1}{2}\dot{w} + \frac{\dot{y}}{2v^2}.$$

Also, putting $\tau = t + s = \frac{1+y}{v}$, we have $\dot{\tau} = \dot{t} + \dot{s}$

$$= \frac{vy - v. \overline{1+y}}{v^2} = \frac{vy + \frac{y\dot{y}}{v} \cdot \overline{1+y}}{v^2} = \frac{v^2 + y + y^2}{v^2} \dot{y} =$$

$$\frac{1+y}{v^2} \dot{y}; \text{ and } \frac{\tau}{v} = \frac{\dot{y}}{v^2}. \text{ Hence, finally, } \frac{\dot{y}}{v^2} = \frac{1}{2} \dot{w} +$$

$\frac{\tau}{2v}$; the fluent of which is $\frac{1}{2} w + \frac{1}{2} \text{ h. l. } \tau$. But this be-

comes $\frac{1}{2} W + \frac{1}{2} \text{ h. l. } T$, or $\frac{\text{fin. } b}{2 \text{ cof.}^2 b} + \frac{1}{2} \text{ h. l. } (\tan. b + \sec. b)$

$$= \frac{\text{fin. } b}{2 \text{ cof.}^2 b} + \frac{1}{2} \text{ h. l. } \tan. 45 + \frac{1}{2} b \text{ when } x = 0, \text{ or when}$$

the exponential fluent $\frac{2x}{a}$ becomes $\frac{1}{2} a$. Theref. the correct

$$\text{fluents are } e^{\frac{2x}{a}} = a + \frac{\text{fin. } b}{\text{cof.}^2 b} - \frac{\text{fin. } z}{\text{cof.}^2 z} + \text{h. l. } \frac{\tan. 45 + \frac{1}{2} b}{\tan. 45 + \frac{1}{2} z}.$$

Mr. Gregory also favoured us with a solution to this question.

XXVI. Or, PRIZE QUESTION 156, answered by the proposer, the late ingenious Mr. John Howard.

Suppose the sphere to descend by its own gravity along any curve *WTBX* (fig. 354, pl. 21.) placed in a plane that is perpendicular to the horizon *QO*; it is required to determine that point *B* where it will quit the curve.

At any small distance above the earth's surface any body descending thereto may be supposed as acted upon by an uniform gravity acting in parallel lines. This being premised, suppose *T* that point in the curve from whence the sphere begins its descent, and let *BD* represent a portion of the parabola *VBD* it will (per gravity) afterwards describe, and whose curvature at *B* will be the same as in the curve *WTBX*. Draw *CB* the common radius, and

BI

BI the common chord, of curvature at B, and produce IB to meet STE, drawn parallel to QO, in E. Also through V draw SVK, perpendicular QO, the axis of the parabola, and from B draw AB its ordinate, and BC the common tangent.

The velocity in the curve at B (in the direction BC) is, by Mechanics, $\sqrt{BE} = \sqrt{AS}$, and the velocity in the parabola at the same point (and in the same direction) is, by projectiles $\sqrt{(VA + \text{sublimity of the parabola})}$, and these velocities being equal by the problem, VS is the sublimity of the parabola, hence, by the parabola, $BE = AS = VS + VA = \frac{1}{2} AK = \frac{1}{2} BO = \frac{1}{2} BI$.

Ex. 1. Suppose VTBX (fig. 355, pl. 21.) a quadrant of a circle in which the sphere begins its descent at any point T below the vertex.

From T let fall TO \perp upon QX and divide it into three equal parts TI, IL, and LO, and through I draw IB \parallel to QO which will cut the curve TX in B the point required, as is plain from the general solution.

Ex. 2. Let the curve VBX (fig. 356, pl. 21.) be the cycloid,

whose equation is $y = z + \sqrt{2ax - xx}$, where $VA = x$, $AE = y$ (E being that point where the ordinate AB meets the semi-circle VEO, whose diam. is the abscissa VO), $z = \text{arch VE}$, $AB = y + z$, and $VO = 2a$. When the quantity x is supposed constant the half chord of curvature will be generally ex-

pressed by $\frac{(1 + \ddot{y})\dot{y}}{-\dot{y}}$, which is also equal to $2x$, by the pre-

sented general solution. Now, in the cycloid $\dot{y} = z + (a - x)$.

$\sqrt{2ax - xx}$ $^{-\frac{1}{2}}$, and $\dot{z} = a \cdot \sqrt{2ax - xx}$ $^{-\frac{1}{2}}$, hence $\dot{y} =$

$\sqrt{2a - x}$ $^{\frac{1}{2}} \cdot x^{-\frac{1}{2}} = \sqrt{2ax - 1 - 1}$ $^{\frac{1}{2}}$, therefore $-\ddot{y} = ax^{-\frac{3}{2}}$.

$\sqrt{2ax - 1 - 1}$ $^{-\frac{1}{2}}$, which values being substituted for \dot{y} and \ddot{y}

in the general equation $\frac{(1 + \ddot{y})\dot{y}}{-\dot{y}} = 2x$, we obtain, by proper

reduction, $x = a$.

Ex. 3: Suppose, in the cycloid, that the sphere begins its descent at the point D (fig. 357, pl. 21.).

Put $AV = x$, draw $DC \parallel$ to OX , and put $VC = n$, then,

supposing every thing else as before, we obtain $\dot{y} = \sqrt{2ax^{-1} - 1}^{\frac{1}{2}}$,

and also $-\ddot{y} = ax^{-2} \cdot \sqrt{2ax^{-1} - 1}^{-\frac{1}{2}}$, whence $\frac{(1 + \dot{y}^2)\dot{y}}{-\ddot{y}} =$

$$\frac{(1 + 2ax^{-1} - 1) \cdot \sqrt{2ax^{-1} - 1}^{\frac{1}{2}}}{ax^{-2} \cdot \sqrt{2ax^{-1} - 1}^{-\frac{1}{2}}} = \frac{2ax^{-1} \cdot \sqrt{2ax^{-1} - 1}^{\frac{1}{2}}}{ax^{-2} \cdot \sqrt{2ax^{-1} - 1}^{-\frac{1}{2}}} =$$

$$\frac{2ax^{-1} \cdot (2ax^{-1} - 1)}{ax^{-2}} = 2 \cdot (2a - x) = 2x - 2n, \text{ per}$$

question, and $x = a + \frac{1}{2}n$, which, when $n = 1$, is $x = a + \frac{1}{2}$ the same as in the Prize Question, Ladies' Diary 1767.

Ex. 4. There is a building in form of a semi-spheroid its altitude being 43, and the circumference of its base 79 feet. It is required to determine how far a marble, descending from its apex, will continue to touch the building, and at what distance it will fall from the foot of it?

Let $VBXC$ (fig. 358, pl. 21.) represent a semi-section of the building passing through its vertex V , and through B draw the ordinate AB . Put $VO = 43 = a$, $VA = x$, $OX = 12.573 = b$, and $AB = y$.

From the general equation we have $\frac{1 + \dot{y}^2}{-\ddot{y}} \dot{y} = 2x$, and in

the ellipse $\dot{y} = (b \div a) \cdot (ax - xx) \cdot \sqrt{2ax - xx}^{-\frac{1}{2}}$, and

(making x constant) $\dot{y} = (b \div a) \times (-x^{\frac{1}{2}} \cdot \sqrt{2ax - xx}^{-\frac{1}{2}}$

$- ax - xx \cdot \sqrt{2ax - xx}^{-\frac{3}{2}} \cdot \sqrt{ax - xx})$, and substituting

for these values of \dot{y} and \ddot{y} we have

$$\frac{x^2 \cdot (b \div a) \cdot (ax - xx) \cdot \overline{2ax - xx}^{-\frac{1}{2}} + (b \div a) \cdot (ax - xx) \cdot \overline{2ax - xx}^{-\frac{1}{2}}}{2x} =$$

$$2x, \text{ or } (b \div a) \cdot x \cdot \overline{-x^2 \cdot 2ax - xx}^{-\frac{1}{2}} - (ax - xx) \cdot \overline{2ax - xx}^{-\frac{1}{2}}$$

$$b^2 x^3 - a^2 x^3 + 3a^2 x^2 - 4ab^2 x^2 + 3a^2 b^2 x + ab^3 x = a^2 b^3. \text{ In}$$

numbers - 169092x³ + 211931184x² + 883669x = 125684927178, hence x will be found equal to 6 nearly; whence $y =$

$$\frac{\sqrt{2ab^2x - b^2x^2}}{a} = 6.406. \text{ Now since the path BD which the}$$

marble afterwards describes is a parabola, we have, by proper reduction XD = 4.657 feet the distance required.

The same, by Philo, of Newcastle-upon-Tyne.

Let AB (fig. 359, pl. 21.) be a curve placed in a plain \perp to the horizon, and suppose the sphere, by the force of gravity, descends from B down the curve and quits it at A.

Draw AP \parallel , and FB \perp to the horizon, continue FB to meet a tangent to the curve, at A, in H; draw BC \parallel to AP, and AO \perp to AH, making AO = the radius of curvature at A, let fall OP \perp to AP. Suppose G to be the vertex of a parabola coinciding with the curve in A, having the same radius of curvature AO, and whose parameter is 4CG (DE being drawn through G \parallel to FH). Then it is evident that a body projected from G, horizontally, with the velocity that would be acquired by falling freely through CG, will have the same velocity and direction at A as the sphere which rolls down the curve from B; therefore both the sphere and the body will move in the parabola GA continued below A. But from the nature of the parabola we have AE =

$$(\overline{CE - CG} \cdot 4CG)^{\frac{1}{2}}, AD = (\overline{CE - CG} \cdot 4CE)^{\frac{1}{2}}, \text{ and } AO$$

$$= 2CE^{\frac{1}{2}} \div CG^{\frac{1}{2}}, \text{ and by the sim. } \triangle s \text{ AED, APO, } (\overline{CE - CG} \cdot$$

$$4CE)^{\frac{1}{2}} : (\overline{CE - CG} \cdot 4CG)^{\frac{1}{2}} :: 2CE^{\frac{1}{2}} \div CG^{\frac{1}{2}} : OP = 2CE.$$

Hence this general rule.

From the nature of the curve, find the ordinate AF, tangent AH, and radius of curvature AO, all in terms of the abscissa

2 B 3

FB

FB and known quantities, then FB will be found from this equation, $AF \times AO = AH \times 2FB$.

Cor. The sphere will never quit the curve if its nature be such, that the ratio of the tangent to the ordinate be always less than that of the radius of curvature to twice the abscissa.

The same, by Mr. John Lowry, Birmingham.

Let APB (fig. 360, pl. 21.) be the given curve placed perpendicular to the plane of the horizon, A the point where the globe begins its descent, and P the point where it quits the curve. Draw the tangent PQ and ordinate FP; then it is evident that the globe will quit the curve when its horizontal velocity, or the velocity in the direction of its ordinate FP, is the greatest possible. Now, by mechanics, the flux. of the curve AP, is to the flux. of its ordinate FP, as the velocity of the globe at P in the direction of the tangent PQ, is to its velocity in the direction of the ordinate FP, therefore $(\dot{FP} \div \dot{AP})$ multiplied by the velocity at P in the direction PQ, is a maximum. But the velocity at P in the direction PQ is as \sqrt{AF} , the height fallen through, therefore $(\dot{FP} \div \dot{AP}) \times \sqrt{AF}$, or its equal $\dot{FP} \times \sqrt{AF} \div \sqrt{(AF^2 + FP^2)}$ must be a maximum. Hence this general rule;

Find the value of \dot{FP} from the equation of the curve in terms of AF and known quantities, substitute the value thus found instead of \dot{FP} in the above expression, then the fluxion of this expression being equated or made equal to nothing, we shall obtain, by reduction, the value of AF, and from thence the point P is readily determined.

The above is applicable to all cases where the curve is referred to a fixed axis, but if the curve be a spiral whose pole is the centre of force, the solution will be considerably different. In this case let SPB (fig. 361 and 362, pl. 21) be the given spiral whose pole is the centre of force S, B the point where the globe begins its descent, and P the point where it quits the curve. Then, since the force of gravity is inversely as the square of the distance from the centre of force, it follows, from the known laws of centripetal forces, that the globe would describe a conic section whose focus is in the centre of force S, was it not prevented from so doing, by moving in the spiral track BPS, in which track it will continue to move till its velocity becomes the same as that in a conic section at the same distance SP.

Now let APR be the conic section which the globe will describe after it quits the spiral, S and J its foci, AR the transverse, and MN the conjugate axis, draw the tangent PC, which will touch both

both the curves at P, for since the velocity is the same in each at that point, the curvature of each must also necessarily be the same; from the focus S on PC demit the \perp SC, join SB, SP, and with the centre S and distance SP describe a circle to cut SB in E. Put SB = a , the transverse axis = $2b$, SP = SE = x , and SC = y ; then, by *Emerson's Centripetal Forces* Prop. xiv. Cor. 1, the velocity in a circle at the point P, is to the velocity in a conic section,

at the same distance, as \sqrt{b} to $\sqrt{2b \pm x}$ (the upper sign being for the hyperbola, and the lower one for the ellipsis): but *ibid.* Prop. v. Cor. 5, the velocity in a circle is reciprocally as the square root of the distance, that is, as $x^{-\frac{1}{2}}$; therefore the velocity in the conic section will be $= \sqrt{(2b \pm x) \div bx}$.—Again, since the velocities are equal at equal distances from the centre of force S, it follows that, the velocity in the spiral at P, will be the same as the velocity at E, acquired by descending along the right line BE; then by *Vince's Fluxions* Art. lxxiii. Ex. 3, the velocity at P is $= \sqrt{(2a - 2x) \div ax}$, therefore $(2b \pm x) \div bx = (2a - 2x) \div ax$, and, by reduction $2b = \mp a$; consequently the transverse axis of the conic section which the globe will describe after it quits the spiral is always equal to the distance between the point where the globe begins its descent and the centre of force. Now to determine the distance SP, by prop. xiv. *Emerson ibid.* the velocity in the spiral is to the velocity in a circle at the distance x , as $\sqrt{yx} : \sqrt{xy}$, hence, it follows, that, if from the nature of the spiral we determine the value of y , we shall obtain the ratio of \sqrt{yx} to \sqrt{xy} in terms of x and known quantities, that is, we shall obtain the ratio of the velocity of the Globe in the spiral, to that in the circle at the same distance in terms of x and known quantities, let this ratio be expressed by that of $n : m$, then by

what is done above, it is evident that $\frac{m}{n\sqrt{x}} = \sqrt{\frac{2b \pm x}{bx}}$, or $\frac{m}{n} = \sqrt{\frac{2b \pm x}{b}}$, or $\frac{m^2}{n^2} = \frac{2b \pm x}{b}$, from which equation x becomes

known, and consequently the point P.

The same, answered by Mr. Richard Elliott, Liverpool.

Suppose BC (fig. 363, pl. 21.) to be the given curve whose vertex is B, and let E be the point where the body quits the curve, after rolling from the point G.—Now it is well known that

that when the body quits the curve, the velocity in the direction of the ordinate DE will be the greatest possible, and it is also well known that the velocity of the body in the direction of curve, is to the velocity in the direction of the ordinate, as the fluxion of BE to the fluxion of DE, also, by the laws of falling bodies, \sqrt{FD} will express the velocity at E, therefore

$BE : DE :: \sqrt{FD} : DE \sqrt{FD} \div BE =$ the velocity of the body in the direction of the ordinate DE, the fluxion of which being made $= 0$, and the equation properly reduced, it is evident that BD will be found, and thence the point E.

Let the curve be the quadrant of a circle, the radius of which call a , put $BF = m$, $BD = x$, and $DE = y$; then, by the equation of the circle, $y^2 = x(2a - x)$, moreover by *Simpson's Fluxions. Art. 142*, the flux. $BE = ax \div \sqrt{2ax - x^2}$, and the flux. of $DE = x(a - x) \div \sqrt{2ax - x^2}$, consequently, from the general proportion above, $(a - x) \sqrt{x - m} \div a$ will be the velocity in the direction DE, which being fluxed and reduced, gives $x^2 + \frac{1}{2}(2a + m)x = \frac{1}{2}(a^2 + 2am)$, from which x may be found.

If $m = 0$, the equation becomes $x^2 + \frac{1}{2}ax = \frac{1}{2}a^2$, from which x is found $= \frac{1}{2}a$; hence it appears that when the body rolls from the vertex, in a circle, that BD is exactly $= \frac{1}{2}$ of the radius.

The same, by Mr. John Surtees, Ep. Wearmouth, Sunderland.

Let AM (fig. 364, pl. 21.) be the given curve, VMN a parabola, V its vertex, F its focus, $BC = \frac{1}{4}$ th of the parameter, and TM a tangent to the parabola and given curve at the point M where the sphere leaves it.—Put $x = AP$, $y = PM$, and $t = TP$, then, by *Emerson's Mechanics, Prop. 38*, and the nature of projectiles, $AP = BM = MF$, by conics, also by *Em. Flux. pa. 218*, the radius of curvature of the parabola at the point M is $(2x \div y) \sqrt{t^2 + y^2}$, which being put equal to the radius of curvature of the given curve at the same point, that which is required may be found.

Sup. AM a circle, rad. $= r$, and vertex A, from whence the sphere descends to M, and there leaves it.

Then $y = \sqrt{2rx - xx}$, $t = (2rx - x^2) \div (r - x)$, and $(2x \div y) \sqrt{t^2 + y^2} = r$, or $\sqrt{3x^2 + 2rx} = r$, hence $x = \frac{1}{2}r$.

The Medal for solving the Mathematical Prize Question, belongs to *Philo, of Newcastle-upon-Tyne*, who is requested to send for it to Mr. Glendinning.

ARTICLE XXXIV.

MATHEMATICAL QUESTIONS.

*To be answered in Number XI.*I. QUESTION 183, *by Mr. John Collins, Schoolmaster, Kensington.*

SUPPOSE a gentleman has 5000*l.* which he puts out at 5 per cent. per annum, compound interest; how much must he expend every day so that principal and interest may be entirely exhausted in 7 years (365 days being a year)?

II. QUESTION 184, *by Mr. John White, Bennington.*

A curious gentleman ordered his gardener to lay out a new garden, in form of a circular segment, to contain 2 acres, and the versed sine to be 4 chains; and as he intends to have a wall quite round it, 7 feet high and 1 foot and a half thick, he wants to know how many solid yards the wall will contain, in order to ascertain the expence: but his gardener being unskilled in calculation requests that some ingenious gentleman will find him the diameter of the circle and the content of the wall?

III. QUESTION 185, *by Merones Minor.*

Required the force of gravity at the earth's surface, in lat. 30° , supposing that, at the equator, to be 1, and the ratio of the equatorial diameter to the axis as 231 to 230?

IV. QUESTION 186, *by Mr. J. Barr, Student in Mr. Robert Wallace's Academy, St. John's Lane, West Smithfield, London.*

If the moon were to revolve about the earth's surface in a circular orbit at the distance of 100000 miles from the earth's centre. It is required to determine the velocity, and periodic time, allowing the semidiameter of the earth to be 21000000 feet?

V.

V. QUESTION 187, by Mr. John Blackwell, *Hungerford.*

On April 28, 1799, in lat. $51^{\circ} 23'$, at eight o'clock, I observed a right-cone standing on a true horizontal plane, the circumference of whose base was 15 feet, around which I applied a cord 36 feet long, and taking both ends in my hand I found it would exactly reach the extremity of the cone's shadow. Query the solidity of the cone?

VI. QUESTION 188, by Mr. J. T. M'Doneld, *Holland, near Wigan.*

To find 3 fractions such that the product of any two of them being subtracted from unity may leave a perfect square?

VII. QUESTION 189, by Mr. Francis, *Hampstead, Middlesex.*

Two ships take their departure from the same port at the same time, the one sailing S. by W. $\frac{1}{2}$ W. the other S. by E. $\frac{1}{2}$ E. from the meridian of the place: after being out $2\frac{1}{2}$ hours they found a rock, (which is known to be 9 leagues from the port, and on the same meridian,) to be in a direct line between them, and that the wind, being N. E. caused a difference of $1\frac{1}{2}$ league per hour in the vessels' sailing. Query their distance from each other and at what rate they sailed?

VIII. QUESTION 190, by Mr. W. Burdon, *Acaster Malbis, York.*

Each side of the base of a triangular pyramid is 5, and its perpendicular altitude 9; which being set upon its base on a plane that makes an angle with the horizon of $33\frac{1}{4}$, will just support itself. Query the position in which it was placed?

IX. QUESTION 191, by Mr. William Murrat, *Boston.*

A merchant ordered his carpenter to make him a crane for the purpose of unloading ships, &c. the length of the upright piece to be 8, and the length of the horizontal piece $4\frac{1}{2}$ feet; now the carpenter, being unskilled in the science of mechanics, thinks, that another piece, reaching from the farther end of the horizontal piece to the lower end of the upright one, will support the horizontal piece with the greatest force possible: but his employer tells him he is wrong; he, therefore, requests the favour of some
learned

learned gentleman to inform him what length the oblique piece ought to be to have the desired effect?

X. QUESTION 192, by Mr. Olinthus Gregory, Cambridge.

The generating circle of a cycloid is 16. Now, if half this cycloid revolve about its base and about its axis respectively, it is required to determine the difference of the greatest cylinders which can be cut out of the solids generated by those revolutions?

XI. QUESTION 193, by the Rev. L. Evans, Royal Academy, Woolwich.

If a conical frustum, the length of which is 20, and the diameters of the greater and less ends 4 and 3 feet respectively, be suspended by the centre of the less end, how many vibrations will it make in a minute, and what will be the distance between its centre of gravity and that of suspension?

XII. QUESTION 194, by Mr. Robert Wallace, Master of the Academy, in St. John's Lane, West Smithfield, London.

Let two points n , m , move from two given positions B , C , with given velocities along two given straight lines, CB , CA , making any given angle with each other at the point C . It is required to determine the curve to which the straight line nm , joining the points n , m , shall always be a tangent?

XIII. QUESTION 195, by Mr. J. T. McDonell.

Given the perimeter of a plane Δ to construct it so, that the line bisecting the vertical \angle , and terminating in the base, may have to the segment of the base, intercepted by the said bisecting line and the \perp , a given ratio, and moreover that the sum of the squares *plus* the rectangle of the sides may be equal to the square of the base.

XIV. QUESTION 196, by Mr. J. H. Swale, Chester.

Given the two sides to construct the plane Δ when the rectangle of the segments of the base made by the \perp has to the square of the \perp a given ratio.

XV.

XV. QUESTION 197, *by Mr. Harris, Carmarthen.*

It is required to determine, by a geometrical construction, the nearest distance between two straight lines given by position, but which are not in the same plane.

XVI. QUESTION 198, *by Mr. O. G. Gregory, Cambridge.*

At what depth, below the surface of the water, must an aperture, 15 inches square, be cut in a sluice door, so that 166·2938889 gallons of water may be discharged through it in a second?

XVII. QUESTION 199, *by Mr. J. Johnson, Birmingham.*

Given the vertical \angle and the length and breadth of a rectangle inscribed in a plane Δ to construct it, when the rectangle contained by the segments of the sides intercepted between their points of contact with the given rectangle and the angular points at the base, may be of a given magnitude.

XVIII. QUESTION 200, *by Mr. W. Peacock, Birmingham.*

In a given Δ to inscribe a rectangle such that its perimeter may be a maximum.

XIX. QUESTION 201, *by Mr. J. Fletcher, Hollinwood.*

As it is well known that the Δ EDF, formed by joining the three points where perpendiculars from the extremities of the base and vertical angle meet the line bisecting that angle and the base respectively, is always similar to the original Δ ABC (see solution to quest. 153), it is evident that another Δ may, by the same operation, be formed from the Δ EDF which will be similar to both the former, another from the last and so on *ad infinitum*, now supposing all these possible Δ s drawn and circles circumscribing them. *Query*, where will be the vertex of the Δ ABC when the sum of all the diameters of these circles is equal to a given line, the diameter HG of the original circumscribing circle being given.

XX. QUESTION 202, *by Mr. Newton Bosworth, Cambridge.*

In the Scholium to Prop. 27, B. I. of the Principia, the following

lowing theorem is given without a demonstration. If a straight line is drawn through any point given by position so as to cut a given conic section in two points, and the distance of the intersections is bisected, the locus of the point of bisection will be another conic section of the same kind with the former, and having its axes parallel to the axes of the former. Query demonstration?

XXI. QUESTION 203, by Mr. Lowry, Birmingham. -

From one of the extremes B of the diameter AB of a given semicircle, whose centre is O, to draw a line BC to terminate in the circumference at C, intersect a chord AD, given by position, in E, and the radius DO in F so that the ratio CE to EF may be a given one.

XXII. QUESTION 204, *by Mr. Lowry.*

From one of the angular points B of a given plane Δ , ABC to draw two straight lines BH, BI to terminate in the opposite side AC at H and I, and intersect a line DE given by position and parallel to AB in G and F so that AH may have to HI a given ratio, and GF be of a given length.

XXIII. QUESTION 205, by A. B.

If there be a series of quantities $a + bx + cx^2 + dx^3 + ex^4 + \&c.$ and the n th differences of $a, b, c, d, \&c.$ be equal among themselves; it is required to find the scale of relation?

XXIV. QUESTION 206, by, James Ivory, Esq.

THEOREM. Let ABC (fig. 367, pl. 21.) be a right angled triangle, right angled at B: draw AD to bisect the angle BAC and bisect BD in E: From the centre A describe the arch BF and draw FG to touch the circle and to meet AE in G: Then will $4AB^2 \times FG = (AB + BD)^2 - (AB - BD)^2$. Query a demonstration.

XXV. QUESTION 207, by Mr. Bulmer.

I have a Fahrenheit's thermometer whose scale is not properly adjusted to the tube, which is cylindrical, for when the bulb is immersed

immersed in water, just freezing, the mercury sinks to 25° , and when in boiling water, it rises to 230° . Now it is required to determine what intermediate degrees of this scale corresponds with the degrees of *temperate*, *summer-heat*, and *blood-heat* marked on a Fahrenheit's truly graduated.

XXVI. QUESTION 208, by A. B.

If a body move in a curve round a centre of force, and if the equation, expressing the relation between the distance of the body from that centre and a perpendicular drawn thence to the tangent,

be $p = \frac{y^n}{a}$; it is required to find the equation of the curve of

aberration. N. B. y = distance of the body from the centre of force, and p = perpendicular thence upon the tangent; a and n being any given numbers.

XXVII. QUESTION 209, by Mr. Lowry.

Let BC, EF be two circles given in magnitude and position, and let A be a given point without them. It is required to draw two straight lines AEF, ABC so that the angle FAC may be of a given magnitude, and so that the sum, or difference of the squares of the chords EF, BC may be equal to a given space.

XXVIII. PRIZE QUESTION 210, by Mr. Wm. Wallace.

Let any number of circles given by position intersect each other at a point P; it is required to draw a straight line from P to meet all the circles in A, B, C, &c. so that the sum of the squares of PA, PB, PC, &c. may be equal to a given space.

ARTICLE XXXV.

Method of discovering the number of negative and impossible Roots, in any equation:

By WILLIAM FRENCH, Esq.

ALGEBRAISTS, who deal in negative or impossible numbers, suppose, that every equation has as many roots as there are units in the highest index of the unknown number in the equation. Consequently, as the roots of an equation are seldom so numerous, they complete the number by roots of other equations, which they call the roots of the given equation, and if the sum of the numbers of the roots, in the two equations, does not amount to the highest index in the given equation, the difference between the highest index and that sum gives a number which they call the number of impossible roots. To discover these negative roots, or impossible roots, is of no use whatever, but they, who are curious after them, may find their number in the easiest way, by the following process.

In any given equation find the number of roots by the process laid down in the second part of my Principles of Algebra. Change the signs of the terms, in which the index of x is an odd number. Find the number of roots, in this transformed equation, which is the number of negative roots to the given equation. Add together the two numbers, thus found, and subtract the sum from the highest index of x , in the given equation, the remainder is the sum of the impossible roots.

EXAMPLES.

Ex. 1. $x^4 - 3x^3 + 75x = 1000.$

$x^4 - 3x^3$ is subtractive, as long as x is less than 3, but $75x$ cannot evidently equal 1000, if x is less than 3, consequently there can be only one root to this equation.

Change the signs of the odd indices, whence the equation becomes

$$x^4 - 3x^3 - 75x = 1000;$$

This equation must have one root, and can have but one.

Hence the given equation has one positive and one negative root. Consequently $4 - 2$ or two is the number of impossible roots.

Ex. 2. $x^3 - 17x^2 + 54x = 350.$

$x^3 - 17x^2$ is subtractive if x is less than 17 and the number to be subtracted is the greatest, when x is equal to $\sqrt{\frac{34}{3}} = \sqrt{11}, \dots$

$= 3, \dots$ Consequently the unknown side cannot be equal to 350, if x is less than 3, and the equation can have only one root.

Change the signs of the odd indices

$$-x^3 - 17x^2 - 54x = 350$$

This is a false equation, consequently there are no negative roots, and subtracting one from three, the number of impossible roots is two.

Ex. 3. $x^3 - 39x^2 + 479x = 1881.$

$x^3 - 39x^2$ is subtractive if x is less than 39 and the number

to be subtracted is the greatest when x is equal to $\sqrt{\frac{39 \times 2}{3}}$

or $\sqrt{\frac{78}{3}}$ or $\sqrt{26} = 5, \dots$ in which case $39x^2 - x^3 = x^2 \times$

$39 - x = x^2 \times 34 = 5 \times 170 = 850$ nearly and $479x = 2395$ and $2395 - 850$ is less than 1881 consequently this equation can have but one root, and equations in this form cannot have a negative root, consequently there are two impossible roots.

Ex. 4. $14937x - 1998x^2 + 80x^3 - x^4 = 5000$

$14937x - 1998x^2$ is additive, if x is less than $\frac{14937}{1998}$ or

7, and, it is evident, that one root is less than unity, and the greatest additive number $14937x - 1998x^2$ is when x is

equal to $\frac{14937}{2 \times 1998}$ or 3, at which time it is far greater than

5000, and the part $80x^3 - x^4$ increasing till x is equal to 60 it is evident, there cannot be another root till x is greater than 7, There must be another root at least and as $-1998x^2 + 80x^3$

$+ 80x^3$ is subtractive till x is equal to 249, ... and is the greatest when $x = 16$, ... if the subtractive part is greater than the additive when x is equal to 16, ... there must be another root less than 16

but $80x^3 - x^4$, in this case, $= 16^3 \times 80 - 16 = 16^3 \times 64$

and $1998x^2 - 14937x = 16 \times 1998 \times 16 - 14937 = 16 \times$

$31968 - 14937 = 16 \times 17031$ but $16 \times 17031 \infty 16^3 \times 64$

$= 16 \times 17031 \infty 16^3 \times 64$ or $16 \times 8 \times 2128, \dots \infty 16^3 \times 8$

or $16 \times 8 \times 8 \times 266, \dots \infty 16^3$. Consequently, the subtractive number being greater than the additive, there must be a root less than 16. Now $80x^3 - x^4$ is the greatest when $x = 60$, in which case it is equal to $60^3 \times 80 - 60$ or $60^3 \times 20$ or

216000×20 which is less than $60 \times 1998 \times 60 - 14937$ or

$60 \times 119880 - 14937$ or 104943×60 consequently, if the increase of $1998x^2 - 14937x$ is greater than that of $80x^3 - x^4$, for every value of x between 16 and 60 this equation could not have another root. To try this, it is to be considered, that the rate of the increase of $80x^3 - x^4$ is the greatest when $3 \times 2 \times 80x$ is equal to $4 \times 3 \times x^3$ or x is equal to 40 in which case $80x^3 -$

$x^4 = x^3 \times 80 - x = 40^3 \times 40 = 2560000$ and $1998x^2 -$

$14937x = 40 \times 1998 \times 40 - 14937 = 40 \times 79920 - 14937$

$= 40 \times 64983 = 2599320$; consequently, as by decreasing x , the part $1998x^2 - 14937x$ decreases much faster than $80x^3 - x^4$, it appears evident almost, that there must be two more roots. Let

$x = 30$ then $80x^3 - x^4 = 30^3 \times 80 - 30 = 27000 \times 80 =$

1350000 and $1998x^2 - 14937x = 30 \times 59940 - 14937 = 30 \times 45003 = 1350090$; Hence $80x^3 - x^4$ increasing faster than $1998x^2 - 14937x$ there must be two roots between 30 and 40 and consequently the given equation is a complete equation, having four roots, or the number of its roots is the same as the number of the highest index. This form of equation is incapable of having negative roots.

Ex. 5. $155x^3 - 20x^2 + 7x^4 - x^5 = 10000$.

$155x^3 - 20x^2$ is additive, if x is less than 7.75, and is greatest when $x = 5, \dots$ in which case, $7x^4 - x^5 = x^4 \times 7 - \dots$

SECRET

1. The first part of the document is a letterhead from the U.S. Department of Justice, Office of the Inspector General, dated 10/10/83.

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2. NAME _____

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1. The Government of the United States, through the Department of the Interior, has been advised by the Bureau of Land Management that the following lands are available for disposal:

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but by $\frac{1}{5}$ part of the whole, (extremely near,) and it is not reasonable to suppose Emerson was not aware of this. Now the perturbing force on the particles in the periphery of any circle perpendicular to AC, being the same throughout, and the boundary of any plane perpendicular to AC being a circle when the perturbing force does not act, will consequently be a circle when the said force does act; and the lines parallel to ACB being lengthened and those parallel to PCQ shortened, the boundary of any plane parallel to APBQ will become an ellipsis, therefore the sea from a sphere will become an oblong spheroid, whose axis is AB. Consequently from the corollary at page 373, of *Emerson's Fluxions* (since the reasoning is the same whether the spheroid be oblate or oblong) we find the gravity at P is nearly: the gravity at A :: $1 + (a-b) \div 5 : 1$. Landen proceeds, "his value of the perturbing force of S, on a particle at D, is also erroneous (being $\frac{2}{3}fy$ instead of fy) and he has omitted the force of S, on a particle at E, in the direction EC." Now this remark is by no means just, for, by the laws of fluids, it is the same thing if a force act in the direction EC, on any particle on the column CP, as if it acted on any particle in the column AC, in the direction CA, and the perturbing force of a particle in the column CP, at the distance y from C, in the direction PC = $\frac{1}{3}fy$, which being considered (as above) to act on the particle at D, we have the whole force at D = $\frac{2}{3}fy + \frac{1}{3}fy = fy$; the same as Emerson makes it. Indeed this way of proceeding is not perfectly just, since there are more particles in CA than in CP, and when y becomes greater than CP the force will only be $\frac{2}{3}fy$; but this force will act on so few particles that the effect will be insignificant, and it would be easy enough to take all in consideration if necessary. Now g being the gravity at A, we have (from above) the gravity

at P = $g + \frac{a-b}{5}g$ nearly, therefore the gravity at E = $\frac{g^x}{a} +$

$\frac{a-b}{5a^2}gx$, therefore Emerson's quotation becomes by this small

correction $\frac{gxy}{a} - fyy = \left(\frac{g}{a} + \frac{a-b}{5a^2}g \right) xy$, which expresses

the fluxion of the equation of the values of the columns that counterbalance each other, the whole fluent of which gives, $a -$

$\frac{f}{g}a^2 = b + \frac{a-b}{5a^2}b^2 = b + \frac{a-b}{5}$, extremely near, there-

fore

fore, $a - b = \frac{4}{3} \frac{fa^2}{g}$ the same as Landen makes it himself. And

I do not think it very strange that Emerson should have contented himself with his approximation, seeing the problem, on which the adjusting the gravity at A and P, is much more difficult than the proposed problem itself would be without such adjustment and that the result in this manner, differs but by $\frac{1}{3}$ part of the whole.

ARTICLE XXXVII.

THREE PROPOSITIONS FROM LAWSON.

(To be answered in No. XI.)

PROP. XXXVII.

LET A and B be two points in the diameter of a circle whose centre is C, and let the rectangle ACB be equal to the square of the semidiameter; bisect AB in D, and raise the perpendicular DM; from the point A draw AF to any point F in the circumference, and FE perpendicular to DM; then I say that the square of AF is equal to twice the rectangle contained by AC and FE.

PROP. XXXVIII.

If any regular figure be circumscribed about a circle, and from any point within the figure there be drawn perpendiculars to all the sides of the figure; the sum of the perpendiculars will be equal to the multiple of the semidiameter of the circle by the number of the sides of the figure.

PROP. XXXIX.

Let there be any number of right lines intersecting in a point, and making all the angles about the point equal, and let any circle pass through the same point; I say the circumference thereof will be divided by the intersecting lines into as many equal parts as there are lines.

ARTICLE XXXVIII.

ON PRESSURE.

By Mr. JOHN LOWRY.

PROBLEM I. Fig. 368, Pl. xxii.

GIVEN the direction of two forces, supporting a body in equilibrio, to find the ratio of those forces.

Let the body D be supported by the two forces DC, DF, acting in the given directions Dc, Df, and let DB, drawn perpendicular to the horizontal plane PQ, represent the force and direction of the body D, and complete the parallelogram BCDF. Then, by the composition and resolution of forces, the force DB is equivalent to the sum of the forces DC, DF; but DF is = BC; therefore the force DC, acting in the direction Dc, is to the force DF, acting in the direction Df, as CD to CB, that is, as the sine of the angle DBC or BDF to the sine of the angle BDC.

Note, the lines CD, DF, DB are supposed to be in the same plane.

PROBLEM II. Fig. 369. Pl. xxii.

Given the position of two planes, supporting a given sphere, to find the pressure against each plane.

Let AE, BE, be the planes given by position; with respect to the horizontal plane PQ, and OAB the given sphere, whose centre is O. Conceive a plane to pass through the points of contact A, B, and at right angles to the horizontal plane PQ; this plane, it is evident, will also pass through O the centre of the given sphere. Draw OAP, OBQ, to meet PQ at P, and Q, and from O demit the perpendicular OC. Then it is evident, that AO is perpendicular to the plane AE, and OB to the plane BE, and since O is the centre of gravity of the supported body, the pressures against the planes AE, BE, will be exerted in the directions OA, OB, and the force of the sphere in the direction OC. But, by the last problem, the force in the direction OA, is to the force in the direction OB, as the sine of the angle QEB to the sine of the angle PEA, therefore the pressure on the plane AE is to the pressure on the plane BE, as the sine of the angle QEB to the sine of the angle PEA; and since the two planes sustain the whole weight of the sphere, the pressure on each may be easily determined.

PROBLEM. III. Fig. 370. Pl. xxii.

If a plane, given in position together with a sphere, given in magnitude and position, support another given sphere, it is required to find the pressure on the sustaining sphere and plane.

Let AE be the plane given, in position DBI the sphere given in magnitude and position, and OAB the supported sphere. Conceive a plane to pass through D, the centre of the sphere DBI, and at right angles to the plane AE, as well as to the horizontal plane PQ, this plane will evidently pass through O, the centre of the sphere OAB, and the points of contact A, B. Draw DV perpendicular to the plane AE, and on DV demit the perpendicular OS; through B, let a plane be conceived to pass, so as to touch the spheres and meet the plane AE at E. Draw OD, which will evidently pass through B, and on AE demit the perpendicular OA. Then it appears, as before, that AO, OD, are the directions of the forces acting against the plane and sphere respectively; hence, if OC be drawn perpendicular to the horizontal plane PEQ, we have by prob. 1, the pressure against the plane AE to the pressure against the sphere, as the sine of the angle DOC to the sine of the angle AOC, that is, by similar triangles, to the sine of the angle AEP.

Now the angle AEP is given, by the problem, and the angle DOC may be easily found by calculation, for VD is given, and $VS = AO$ is also given, therefore DS is given, whence DO and the angle DOS become known, and consequently $(\angle DOS + \text{comp. } \angle AOV = \text{AEP or})$ the angle DOC is known.

PROBLEM IV. Fig. 371, Pl. xxii.

Let two spheres, given in magnitude and position, support a third given sphere, it is required to find the pressure against each of the sustaining spheres.

Let AR, DB be the two given sustaining spheres, placed at a given distance from each other, on the horizontal plane PQ, and AOB the other given sphere. Conceive a plane, at right angles to the horizontal plane PQ, to pass through the centres D and R, this plane will, it is evident, pass through O, the centre of the other sphere and likewise through the points of contact A and B. Draw OR, OD, which will pass through A and B respectively, and on PQ drop the perpendicular OC. Then it appears, as in the preceding problems, that the pressure against the sphere AR is to the pressure against the sphere BD, as the sine of the angle DOC to the sine of the angle AOC.

To determine these angles by calculation, draw RS perpendicular, and DS parallel, to the horizontal plane PQ, and join RD. Then in the right angled triangle RSD, RD is given, consequently the angle RDS may be found; and in the triangle DOR, the three sides are given, therefore the angles ROD, RDO may be found, and from thence the angle SDO, and its complement the angle DOC; hence the angle AOC becomes known.

PROBLEM V. Fig. 372, Pl. xxii.

Given the directions of three forces, supporting a body, in equilibrio, to find the ratio of those forces.

Let the forces AD, AB, AC acting in the directions AD, AI, AH respectively, sustain a body at A acting in the direction AE perpendicular to the horizontal plane RQ. Let AE represent the force of the body A, and conceive a plane to pass through AD, AE and intersect the plane which passes through AI, AH, in the straight line AF, and complete the parallelograms EDAF, FBAC. Then, by the composition of forces, the sum of the forces AD, AF is equivalent to the force AE, and the sum of the forces AB, AC is equivalent to the force AF; hence the sum of the forces AD, AB, AC acting in the directions AD, AI, AH is equivalent to the force AE.

Conceive a plane to pass through the point A at right angles to AE, and on it demit the perpendiculars Bb, Cc, Ff, and join Ab, Ac, Af. Then, by prob. 1,
 $AD : AF :: \sin \angle EAF \text{ (or } AFf) : \sin \angle EAD$; but by Trigonom.
 $AF = Af \div \sin \angle AFf$ (radius being supposed = 1); therefore,
 $Af = AD \times \sin \angle EAD$. Again

$Af : Ab :: \sin \angle bAc \text{ (supp. } Abf) : \sin \angle fAc \text{ (} Afb)$, and
 $Ab = AB \times \sin \angle EAB \text{ (cosin. } BAb)$. Therefore, we have
 $AD \times \sin \angle EAD : AB \times \sin \angle EAB :: \sin \angle bAc : \sin \angle fAc$; hence
 $AD : AB :: \sin \angle bAc \div \sin \angle EAD : \sin \angle fAc \div \sin \angle EAB$.

And, in the same way, it appears, that
 $AD : AC :: \sin \angle bAc \div \sin \angle EAD : \sin \angle fAb \div \sin \angle EAC$. Hence,
 the three forces } $\frac{\sin \angle bAc}{\sin \angle EAD}, \frac{\sin \angle fAc}{\sin \angle EAB}$, and $\frac{\sin \angle fAb}{\sin \angle EAC}$ respec.
 AD, AB, AC are as

COROLLARY. Draw DM and EN parallel to AB to meet EM and DN, drawn parallel to AC, at M and N, and join MB, NC; then the figure ACFBDNEM is a parallelopiped whose three sides are AD, AB, and AC and whose diagonal is AE. Hence this

THEOREM.

Three forces, acting in the direction and having the proportion of the sides of a parallelepiped, are equivalent to a force, acting in the direction and having the proportion of the diagonal.

This theorem is of very extensive use, in the science of Mechanics.

PROBLEM VI.

Given the position of three planes, supporting a given sphere, to find the pressure against each plane.

Conceive three planes to be drawn parallel to the given ones, at a distance therefrom equal to the radius of the given sphere, then it is evident, that the intersection of these planes will be the centre of the sphere, when it rests in equilibrio, and, if perpendiculars be demitted from that point, on the given planes, the pressures, against the respective planes, will be exerted in the directions of these perpendiculars. Hence the directions of three forces being given, the forces themselves may be determined by the last problem.

The calculation will be extremely easy, for if a plane be conceived to pass through the centre of the sphere and parallel to the horizon, intersecting the given planes, the angles of intersection, which are given, will be equal to the respective angles of direction.

PROBLEM VII. *Fig. 373, Pl. xxii.*

If two planes, given in position, together with a sphere, given in magnitude and position, support another given sphere, it is required to find the pressure against each.

Let O be the centre of the given sphere, resting on the horizontal plane. Conceive two planes to be drawn parallel to the given planes, at a distance therefrom equal to the radius of the supported sphere. Let the straight line EF be the intersection of these planes which will be given in position, and in which the centre E of the supported sphere will be situated. Now the point O, and the distance OE (equal to the sum of the radii of the given spheres) being given, the point E is given, hence if EO be joined and EA, ED be drawn perpendicular to the given planes, the pressures will be exerted in the directions EO, EA, ED, and consequently their ratios may be determined by prob. VI.

The method of calculation is so evident from prob. V and VI. that it is needless to repeat it.

PROB-

PROBLEM VIII. Fig. 374, Pl. xxii.

Let a plane, given in position, together with two spheres, given in magnitude and position, support another given sphere, it is required to find the pressure against each.

Let O and Q be the centres of the given spheres, and CL the plane given in position. Conceive the plane FI to be drawn parallel to the given one, at a distance therefrom equal to the given radius of the supported sphere. The centre E of this sphere will evidently be situated in the plane FI , and since the distances QE , OE are given, (equal to the sum of the respective radii), the point E is given; hence, if EC be drawn perpendicular to the plane CL ; EC , EA and EB will be the directions of the pressures against the plane CL , the sphere whose centre is Q , and the sphere whose centre is O respectively.

The calculation is easily made out from what is done above.

PROBLEM IX. Fig. 375, Pl. xxii.

Let there be three spheres, given in position and magnitude, supporting a fourth given sphere, it is required to find the pressure against each.

Let O , Q , and G be the centres of the spheres given in position, and E the centre of the other sphere. Then the distances QE , OE and GE are given, being equal to the radius of the sphere whose centre is E , together with the radii of the spheres whose centres are O , Q , and G respectively, therefore, the point E may be determined, and so may the directions OE , QE , and GE of the pressures; whence, and from problem VI, the pressures themselves, become known.

Since O , Q , and G are given points and OE , QF and GE given distances, the point E may be determined by calculation, as in the second solution to question 99 of the *REPOSITORY*, and thence the directions may be found by a single operation in Trigonometry.

When the three spheres, given in position, touch each other, the problem is the same as question 329 *LADIES' DIARY*, and which is left unanswered in that work. The only solution, that I know of, is that given by the late ingenious Mr. *REUBEN BURROW*, in his *DIARY* for 1779, but the method of calculation thence is by no means obvious.

ARTICLE XXXIX.

To the Editor of the Mathematical Repository.

SIR,

AS the following Problem will be found of some service, in the solution of Problems, which, when done algebraically, rise to equations of three or four dimensions; I presume, that this advantage may apologize for its insertion in the Repository. The Problem may be considered as a small addition to the Lemmas of NEWTON in his *Arithmetica Universalis*.

I am, Sir,

Your most obedient,

London, March 10, 1800.

ROBERT WALLACE.

PROBLEM. Fig. 376, Pl. xxii.

Given one of the angles at the base ABC, the difference between the base and one of the sides, $BC - AB$, and one of the segments of the other side, AO or OC, made by the line BO drawn from the angular point B to make given angles AOB, BOC with AC at the point of intersection O, to construct the plane triangle.

CASE I. When AO is given. Make $CD = DF = BC - AB$, the given difference, and the $\angle CDF = \angle ABC =$ the given angle at the base; through F draw $FG \parallel$ to CD, and make the $\angle DFM =$ the given angle AOB; draw $CL \parallel$ to FM meeting FG produced in L, and $CH \parallel$ to FD meeting MF, GF produced in H and P respectively. Take f a 4th proportional to AO, HC, and CD, these being all given, and to FL, FP and f find a 4th proportional m ; now through C draw CMG, so that GM, the part intercepted by GE, FM, may be $=$ to m (vide Newton's *Arithmetica Universalis*, Archimedes' Lemmas, or Pappus' *Mathematical Collections*.) and through D draw DA \parallel to CF meeting CG produced in A, also draw AB \parallel to FD meeting CD produced in B; so shall ABC be the triangle required.

Because the \triangle s ABD, FDC are evidently similar, and $DF = CD$, AB will be $=$ to BD, and $BC - AB = CD$, and also the $\angle ABC = \angle FDC =$ the given \angle at the base.

Make the $\angle AOB = \angle END = \angle DFM =$ the given \angle at the point of intersection. Then by sim. \triangle s

$CD : AO :: FE : EN$; but $FE : EN :: ME : ED$; therefore
ME =

$ME = CD \times ED \div AO$, and theref. $HC : MC :: FE : CD \times ED \div AO$. Hence

$f(CD \times CH \div AO) : CM :: FE : ED$. Moreover

$FE : ED :: GE : CE$, and $GE : EC :: GF : FP$; therefore

$f \times FP = CM \times GF$: but $GM : MC :: GF : FL$. Consequently

$GM \times FL = CM \times GF = f \times FP$, and

$GM = f \times FP \div FL = m$. *Q. E. D.*

CASE II. When CO is given. Make $CD = DF =$ the given difference as before, and draw $FG \parallel$ to CD . Make the $\angle GFM =$ the $\angle COB =$ the given \angle at the point of intersection, and draw $CS \parallel$ to FM meeting FD produced in S . Take a a 4th proportional to CO , TC , and CD , T being the point where FM meets CD produced, and b a 4th proportional to SE , a , and CD . Through C draw $CEMG$ so that ME , the part intercepted between FT , FD , may be $=$ to b , and through D draw $DA \parallel$ to FC meeting CG produced in B , and ABC shall be the triangle required.

Let lines be drawn as before; then because

the $\angle CDN = \angle GFM$, $TC \times CD \div MC = CN$; also

$CN : FE :: CO : CD$, therefore $FE \times CO \div CD = CN$. Hence

$MC : CD :: TC \times CD : FE \times CO$: but $a = CT \times CD \div CO$; theref.

$MC \times FE = a \times CD$. Again $SE : FE :: MC : ME$, theref.

$SE \times ME = FE \times MC = a \times CD$, or $ME = a \times CD \div SE = b$. *Q. E. D.*

Corollary. In Case I. when $CD = 9$, $AO = 18$, $\angle AOB = 28^\circ 30'$, and $\angle ABC = 90^\circ$, the Prob. becoms the same as quest. 3rd. Ladies' Diary, 1797. In this case produce BA till $BX = BC$, and in CH produced take $CY = BC$, and join XY ; $BCXY$ shall be the square required.

The 6th quest. of the Gent. Diary, for the present year, is also a particular case of this Problem.

ARTICLE XL.

To the Editor of the Mathematical Repository,

SIR,

IN Number IX. of your valuable work, are inserted some instances of the mode of search after negatives and impossibles, which I recommend to persons who are curious after those matters. In the third example is an error in the calculation, arising probably from the little interest I take in such subjects, which overthrows the

the result there stated. The instance was taken from BARON MASERES' "Tracts on the resolution of affected algebraic equations, by various methods of approximation," among which Tracts is one, by myself, on the number of negative and impossible roots, but without any instances.

The error in my calculation consists in making $39x^2 - x^3$ the greatest possible when x is equal to $\sqrt{\frac{39 \times 2}{3}}$: but $39x^2 - x^3$

is the greatest possible when $2 \times 39 \times x$ is equal to 31^2 or x is equal to 26. Hence the true reasoning upon the proposed equation is as follows:—

$$x^3 - 39x^2 + 479x = 1881.$$

The unknown side must be less than $479x$, if x is less than 39: if x is equal to 39, the unknown side is equal to $479x$, which is much greater than 1881, and, if x is greater than 39, the unknown side is increased; consequently each root of this equation must be less than 39.

Also, since the greatest subtractive value of $39x^2 - x^3$ is 8788, when x is equal to 26, if in this case the unknown side is greater than 1881, a number greater than 26 cannot make the unknown side equal to 1881. But if x is equal to 26, then $39x^2 - x^3$ is equal to $x^2 \times (39 - 26)$ or $13x^2$ and $479x - 13x^2$ is equal to $x \times (479 - 13x)$ or $x \times (479 - 169)$ or 310×13 , which is evidently greater than 1881: consequently each of the roots must be less than 26.

Now, the unknown side of equations, in this form, first increase from some indefinitely small value of x and then diminish and afterwards increase, or they constantly increase: in the latter case, they have only one root, in the former case, they may have one, two, or three roots, and this number is seen by trying whether the increase of $479x$ is equal to that of $39x^2 - x^3$ or 479 is equal to $78x - 3x^2$, or $159, \dots = 26x - x^2$, or $\sqrt{169 - 159, \dots} = 13 \propto x$, or $3, \dots = 18 \propto x$. Hence, the increase of $479x$, being equal to that of $39x^2 - x^3$, when x is equal to 9, ..., or 16, ..., there can be only two roots, if one root is equal to either 9, ..., or 16, But the root of the proposed equation cannot be either 9, ..., or 16, ... since it does not admit of a decimal for the root; consequently, if there is one root less than 9, ... there must be three roots to the equation*: unless that root is less than or equal to

* See my principles of algebra, on the true theory of equations, established on Mathematical demonstration. Part the second. Pages 50, 60, 70.

$$\text{to } 13 = \frac{2\sqrt{39^2 - 3 \times 479}}{3}, \text{ or } 13 = \frac{2\sqrt{1521 - 1437}}{3} \text{ or } 13$$

$$= \frac{2 \times 9, \dots}{3}, \text{ or } 6, \dots$$

Hence I try 9, as the nearest number to 9,.... and I find $x^2 \times (39 - x)$ to be $x^2 \times 30$ and $479x - 30x^2$ or $x \times (479 - 270)$ or 9×209 to be 1881. Hence the equation has three roots.

I have given the method at large, more to exemplify my general rule, than as the one I should have adopted, if this example had been proposed to me to find the roots themselves. For I should have seen at once, that each root was greater than 1. This would have led me to try 10, as an easy number; and finding the result to be very nearly equal to, though greater, than 1881, and that the increase of $479x$ is less than, though very nearly, equal to that of $39x^2 - x^3$, I should conclude at once, that there must be two roots nearly equal to 10, the one greater and the other less than 10. Of course, I try 9, which succeeds: then 11 which also succeeds; consequently the third root is 19.

In fact, all equations of the third class, in the simplest form, are resolved with great ease, if one of the roots is a whole number, equal to or less than 10.

W. FRENDE

Inner Temple, London, 20th July, 1800.

ARTICLE XLI.

GEOMETRICAL SECTIONS.

By Mr. JOHN LOWRY.

PROBLEM I.

TO or from two given right lines, to add or cut off two others, having a given ratio, or to add the one and cut off the other, so that the ratio, the rectangle, the sum of the squares, or the difference of the squares of the two lines thus, compounded, may be given.

CASE I. When the ratio is given. . *Fig. 377, 378, 379, Pl. xxii.*
To or from two right lines AB, CD, to add as in fig. 377, or cut off as in fig. 378, two others BE, CE having a given

the other, as in fig. 394, so that AE may have to DE a given ratio, and so that the sum of the squares of BE, CE, may be equal to a given square.

On AB take IA to CD in the given ratio of AE to ED, and draw IQ perpendicular to IA and equal to CD; draw AQ, and parallel thereto draw IC, to which, from B, apply BC = the side of the given square; through C let ECD be drawn parallel to IQ to meet AQ, produced if necessary, at D, and it is done.

For, by reason of the parallels, $AE : DE :: AI : IQ (= CD)$, that is, in the given ratio; also since the $\angle DEA$ is a right angle, $EB^2 + EC^2 = BC^2 =$ the given sum of the squares, by construction.

CASE III. When the difference of the squares is given.

Fig. 395, 396, 397, Pl. xxii. To or from two given right lines AB, CD, to add, as in fig. 395, or cut off, as in fig. 396, two others BE, CE, (or to add the one and cut off the other as in fig. 397, so that AE may have to DE a given ratio, and so that the difference of the squares of BE, CE may be equal to a given square.

Perpendicular to AB, draw BC = the side of the given square, and take AI to CD in the given ratio of AE to DE; draw CI, and parallel thereto AD, to which, from I, apply IQ = CD; through C let DCE be drawn parallel to IQ, to meet AB, produced if necessary, at E, and it is done.

For, by parallels, $AE : DE :: AI : IQ (= CD)$, that is, in the given ratio, and because the $\angle CBE$ is a right angle, $CE^2 - EB^2 = CB^2 =$ the given difference of the squares.

PROBLEM III.

To or from two given right lines, to add or cut off two others, or to add the one and cut off the other, so that one of the lines, thus compounded, may have to the part added to, or cut off from the other, a given ratio, and so that the ratio, rectangle, the sum of the squares, or the difference of the squares of the other compounded line, and the part added to, or cut off from the first line may be given.

CASE I. When the ratio is given. Fig. 398, 399, 400, Pl. xxii. To or from two given right lines AB, CD, to add as, in fig. 398, or cut off, as in fig. 399, two others BE, CE, or to add the one and cut off the other, as in fig. 400, so that the ratio of AE to CE, and also the ratio of DE to BE may be given.

Perpen.

Perpendicular to AB let BF be drawn, and let AB to BQ be taken in the given ratio of AE to CE ; draw IQ parallel to AB , and take BQ to QI in the given ratio of DE to BE ; draw AQ , BI , and take $QF = CD$, and from F draw FD parallel to AQ to meet BI , produced if necessary, at D ; through D draw DCE parallel to FB to meet AQ , and AB , produced if necessary, at C and E , and it is done.

For, by reason of the parallels, $AE : CE :: AB : BQ$, and $DE : BE :: BQ : QI$, that is, in the given ratios, by construction.

CASE II. When the rectangle is given. Fig. 401, 402, 403,

Pl. 22. To or from two given right lines AB , CD , to add, as in fig. 401, or cut off, as in fig. 402, two others, BE , CE , or to add the one and cut off the other, as in fig. 403, so that AE may have to CE a given ratio, and so that the rectangle $BE \times DE$ may be equal to a given square.

Perpendicular to FE , the side of the given square, let the indefinite right lines AED , FG be drawn, and take $FG = DC$, and $FH = AB$; divide GH at I , so that HI may have IG in the given ratio of AE to CE , and draw EI to meet a semicircle described on HG at Q ; draw also GQC , HQA , and parallel thereto FD , FB respectively, and it is done.

For, by reason of the parallels, and construction $AE : CE :: HI : IG$, that is, in the given ratio, and $AB (= FH)$ and $DC (= FG)$, are the given lines; also the $\angle AFD = AQC =$ a right angle; therefore, Eu. 8, VI. $DE \times BE = EF^2 =$ the given square.

The above suggests a method different from any I have yet seen, of adding a line to a given line, so that the rectangle under the whole compounded line and the part added may be equal to a given square.

Perpendicular to AD , (fig. 404, pl. 22.), the side of the given square, draw $AB =$ the given line, and also the indefinite line GDF ; from D draw DI to the middle of AB , and take $IC = AI$ or IB , and draw ACF ; then DF is the part required to be added.

For, draw BCE and parallel thereto AG ; then the $\angle ACB$ is evidently a right angle, therefore, by parallels, the $\angle GAF$ will be a right angle; hence $GD \times DF = DA^2$, but since $AI = IB$, $ED = DF$ and $AB = GE$; therefore $(AB + DF) \cdot DF = GD \times DF = DA^2 =$ the given square.

CASE III. When the sum of the squares is given. Fig. 405,

406, 407, Pl. 22. To or from two given right lines AB , CD , to add, as in fig. 405, or cut off, as in fig. 406, two others BE , CE , or to add the one and cut off the other, as in fig. 407, so that AE may have to CE a given

ratio, and so that the sum of the squares of BE, DE may be equal to a given square.

Perpendicular to AB let BI be drawn, and take AB to BI in the given ratio of AE to CE, take also IQ = CD, and draw QD parallel to the line joining AI; from B to QD apply BD = to the side of the given square, and draw DCE parallel to QB to meet AI, AB, produced if necessary, at C and E, and it is done.

For, by reason of the parallels, $AE:CE::AB:BI$, that is, in the given ratio, and AB and CD (= IQ), are the given lines; also because of the right angle at E, $BE^2 + ED^2 = BD^2 =$ the given sum of the squares.

CASE IV. When the difference of the squares is given.

Fig. 408, 409, 410. Pl. 22. To or from two given right lines AB, CD, to add, as in fig. 408, or cut off, as in fig. 409, two others BE, EC, or to add the one and cut off the other, as in fig. 410, so that AE may have to CE a given ratio, and so that the difference of the squares of DE, BE may be equal to a given square.

Perpendicular to AB take BD = to the side of the given square, and take also AI to CD in the given ratio of AE to CE; join ID, to which, from A, apply AQ = CD, and from D draw DCE parallel to IQ to meet AC, drawn parallel to ID, at C, and AB, produced if necessary, at E, and it is done.

For, by parallels $AE:CE::AI:AQ (= CD)$, that is, in the given ratio, and AB and CD (= AQ) are the given lines, also $DE^2 - BE^2 = BC^2 =$ the given difference of the squares.

(To be continued.)

ARTICLE XLII.

Demonstrations of Lawson's Propositions proposed in

ARTICLES XIV. AND XXV.

PROP. XXXI. Fig. 411, 412, 413, Pl. 22.

Demonstrated by Messrs. Campbell and Lowry. Fig. 411, 413-

BY hyp. the rect. EAF = AB^2 = rect. CAD by Eu. 36, III. theref. $AC:AE::AF:AD$, and \therefore the $\triangle ACE, ADF$, are equiangular. Hence the $\angle AEC = ADF = HGE$, theref. GH is \parallel to AE.

Q. E. D.

The same by Mr. James Cunliffe, Bolton-le-Moors. Fig. 411, 412.

Since $AF \cdot AE = AB^2$, and $AB^2 = AD \cdot AC$, theref. $AF \cdot AE = AD \cdot AC$;

Whence $AF : AD :: AC : AE$, Euc. 16, VI, theref. $\angle AFD = \angle ACE$, Euc. 6, VI.

But the figure DCGH is a trapezium inscribed in the circle, theref. the $\angle DCG + DHG =$ two right angles $= \angle DCG + ACE$, by Sim. Geo. 17, III;

whence $\angle DHG = ACE = AFD$, and \therefore by Eu. 29, I, GH is \parallel to AE.

Q. E. D.

The same by Mr. J. Harris, Caermarthen, Fig. 413.

Because, the rect. $EAF = AB^2 =$ rect. CAD , the points E, F, C, D, are in a circle,

therefore $\angle AFD = ACE = GCD$; but $GCD = GHD$, theref. $\angle AFD = GHD$;

wherefore GH is parallel to AF.

Q. E. D.

PROP. XXXII. Fig. 414, 415, 416, Pl. 22.

Demonstrated by Messrs. Campbell, Cunliffe, Harris, and Lowry.

By hyp. the rect. $FAG =$ rect. $BAE =$ rect. CAD , by Euc. 35, III; theref. $FA : CA :: DA : GA$, and \therefore the $\triangle s$ FAC, GAD are equiangular. Hence the $\angle CFE = CDK = CHK$, theref. HK is \parallel to AB. *Q. E. D.*

PROP. XXXIII. Fig. 417, 418, 419, 420, Pl. 22, 23.

Demonstrated by Mr. Colin Campbell, Liverpool. Fig. 417.

Draw $AN = AE$ to meet EB in N, and from A demit the \perp AL upon NE.

Then $AB^2 = AG^2 + BG^2 + 2BGL$ (Euc. 12, II.) $= AG^2 + BG^2 + BG \cdot (GE + GN) = AG^2 + BGN + GBE = AN^2 + NGE + BGN + GBE$ (Sim. Eu. Dat. p. 427, lem.)

But $AB^2 = AC^2 + BD^2$, by hyp. $= EAF + GBE$, Euc. 36, III; hence $AE^2 + NG \cdot BE + GB \cdot BE = EA \cdot AF + GB \cdot BE = AE^2 + AE \cdot EF + GB \cdot BE$;

therefore $NG \cdot BE = AE \cdot EF$; conseq. $NG : AN :: EF : BE :: GH : HB$. wherefore, since $\angle ANG = \angle GEF = \angle GHB$, (Euc. 28, III), the $\triangle s$ ANG, GHB are equiangular, and the $\angle NGA = \angle HGB$ (Euc. 6, VI.);

theref. $\angle HGB + \angle EGH =$ two right angles $= \angle NGA + \angle EGH$; conf. A, G, H are in a right line (Eu. 14, I.). *Q. E. D.*

The same by Mr. James Cunliffe. Fig. 418.

In AB take a point n , such that $Bn \times BA = BD^2$, and draw the lines nF , nH , AH , and GH . By hypothesis

$AB^2 = An \times AB + Bn \times AB = AC^2 + BD^2$, therf. $An \times AB = AC^2$: but

$AF \times AE = AC^2$, and $BG \times BE = BH \times BF = BD^2$, Euc. 36, III; therf. $AF \times AE = An \times AB$, and therf. $AF:AB::An:AE$, Eu. 16, VI; hence $\angle AnF = AEB = FEG$, Eu. 6, VI. whence $\angle FnB = FHG =$ supplement of $\angle E$.

Moreover, by Constr. $Bn \times BA = BD^2 = BH \times BF$, Euc. 36, III. whence $BF:BA::Bn:BH$, Eu. 16, VI; therf. $\angle AHB = FnB = FHG$, and therf. AGH is a straight line, by Euc. 15, I. *Q. E. D.*

The same by Mr. John Harris. Fig. 419. 420. Pl. 23.

Let the diameter KIL be drawn perpendicular to AB meeting the circle in K and I , and AB in L . Then by hypothesis, $AC^2 + BD^2 = BA^2$: but, by Prop. XI.

$AC^2 = ILK + AL^2$, and $BD^2 = ILX + LB^2$, therefore $2ILK + AL^2 + BL^2 = AB^2 = AL^2 + BL^2 + 2ALB$, i.e. the rect. $ILK = ALB$.

Hence, the rect. $LAB (= ALB + AL^2) = AC^2 = EAF$, and the rect. $LBA (= ALB + BL^2) = BD^2 = EBG$; therf. the points L, B, E, F are in a circle, and so are also the points E, G, A, L ;

conseq. $\angle EFH = ELA$ or $ELB = EGA$: but $\angle EGH + EFH =$ two right angles;

therf. $\angle EGH + EGA =$ two right angles, and $\therefore AGH$ is a right line. *Q. E. D.*

The same by Mr. John Lowry. Fig. 418.

In AB take the point n such that $AB \cdot An = AC^2$, then $DB^2 = AB \cdot Bn = BG \cdot BE$, therefore, joining En , (intersecting AG in K .) the Δ s AGB, BnE are equiangular, and, of course, so are the Δ s AKn, EKG , therefore the $\angle AGE (= AnE = AFB) = BGH$, as is shewn in the demonstration to Prop. 31; therefore the points A, G, H , are in a right line. *Q. E. D.*

PROP. XXXIV. *Fig. 421, 422, 423, Pl. 23.*

Demonstrated by Mr. Campbell. Fig. 421.

Bisect CD in I and draw FI . Then by hypothesis $CAD = AB^2 + CBD = AB^2 (AC + CB) + CBD = BAC + CB \cdot AD$, consequently

$CA \cdot (AD - AB) = CA \cdot BD, = CB \cdot AD$, therefore
 $AC : CB :: AD : BD$, and therefore, by Prop. I. the rect. $ABI =$
 CBD . Hence

$CAD = EAF = AB^2 + ABI = AB \cdot (AB + BI) = BAI$, conseq.
 $AF : AI :: AB : AE$, therof. the Δ s FAI , ABE , having $\angle A$
 common, are equiangular, and therefore the $\angle AEB =$
 AIF . Moreover, because

$AB \cdot BI = CB \cdot BD = HB \cdot BF$, $AB : BH :: BF : BI$; therof. the Δ s
 ABH , FBI , in which the $\angle ABH = FBI$, are also
 equiangular, and $AHB = AIF$.

Wherefore, seeing $\angle GEF = FHG$, $\angle GEF + AEB = 2$ right
 angles $= FHG + AHB$.

Conseq. the points A , H , G are in the same right line. *Q. E. D.*

The same, by Mr. James Cunliffe, Fig. 422.

Join the point A with O the centre of the circle, and draw
 BQ , and $OI \perp$ to AO and AD respectively; produce BQ till it
 cuts the circle again in T and t , and draw AT , as also the radius
 OT .

The line Tt is bisected in the point Q by Euc. 3, III. and by
 Euc. 12, II.

$AT^2 = AB^2 + TB^2 + 2BQ \cdot TB = AB^2 + TB \cdot (TB + 2BQ)$
 $= AB^2 + TB \cdot Bt$, (Euc. 35, III.) $= AB^2 + CB \cdot BD$: but, by
 hypothesis

$AB^2 + CB \cdot BD = AC \times AD$, therof. $AT^2 = AC \times AD$; therefore
 AT is a tangent to the circle at the point T , by Euc. 36, III.

Now ATO is a right angle, therefore the Δ s ATO , AQT ,
 are similar; and it is also evident that the Δ s AIO , AQB , are
 similar, therefore,

$AQ : AT :: AT : AQ$, and $AQ : AB :: AI : AO$, and therof. Euc. 16, VI.

$AB \cdot AI = AQ \cdot AO = AT^2 = AB^2 + CB \cdot BD$; wherefore,

$AB \cdot BI = AB \cdot (AI - AB) = AB \cdot AI - AB^2 = CB \cdot BD =$
 $BG \cdot GF$, whence

$AB : BG :: GF : BI$; but $\angle ABG = FBI$, therof. Euc. 6, VI. $\angle BGA$
 $= BIF$.

Again, since $AB \cdot AI = AT^2 = AE \cdot AF$, therof. Euc. 16, VI.

$AB : AE :: AF : AI$; therof. $\angle AEB = AIF = BIF = BGA$.

Moreover $\angle FEH = FGH$, therof. $\angle FEH + AEB = FGH + BGA$;
 but $\angle FEH + AEB = 2$ right angles, therof. $\angle FGH + BGA$
 $= 2$ right angles;

therefore the points A , G , H , are in a right line. *Q. E. D.*

The same, by Mr. John Harris, Fig. 423.

Through A draw the diam. AK , and \perp to it draw BQ cutting
 2 E 3 it

it in Q and the circle T and t ; draw also EQR, HQI, and join FR, GL (meeting Aq in P,) AI, AT, and HE.

Then rect. CAD = $AB^2 + CBD = AB^2 + TBt = AB^2 + TQ^2 - BQ^2 = AQ^2 + TQ^2 = AT^2$,

therefore AT touches the circle at the point T; hence, by Prop. XV, AK: Aq :: KQ: Qq, and, by Conv. Prop. VII. FR is parallel to Tt; therefore

$\angle HBQ = HFR = HEQ (= HER)$, conseq. the points H, E, B, Q are in a circle; hence

$\angle BQI = HEG (= HEB) = HIG$, therefore IG is parallel to Tt, IP = PG: but

$\angle PAI = PAH$, by Prop. IV; theref. AHG is a right line. Q. E. D.

The same, by Mr. John Lowry. Fig. 421.

Draw FI to meet AD at I and make the $\angle FID = FEG$, then the $\angle AIF + GEF = 2$ right angles, therefore the points E, B, I, F, are in the circumf. of the same circle; hence the rect. BAI = EAF = CBD =, by hypothesis, $AB^2 + CBD$, therefore taking away the square of AB from the first and last of these equals, the rect. ABI = CBD, that is = HBF; therefore the points A, H, I, F are in the circumf. of the same circle, conseq. $\angle AHF = AIF$: but $\angle GHF = GEF$; theref. $\angle AHF + GHF = AIF + GEF = 2$ right angles, and theref. the points A, H, G are in a right line. Q. E. D.

PROP. XXXV. *Fig. 424, 425, 426, Pl. 23.*

Demonstrated by Mr. Colin Campbell.

Join GE in Fig. 424, and HE in Fig. 425. Then $BA \cdot AE = AC^2 = GA \cdot AH$, and $EB \cdot BF = BD^2 = GB \cdot BK = LB \cdot BK$, therefore in

CASE I. When the points E and F are taken between the points A and B.

$BA : AG :: AH : AE$, conseq. the Δ s ABH, AGE are sim. and $\angle AHB = AEG$:

but $\angle AEG + GEB = 2$ right angles = $AHB + GKL$; hence $\angle GEB = GKL$. And

$BG : BF :: BE : BK$, theref. the Δ s BGE, BKF are sim. and $\angle GEB = BKF$.

Whence $\angle GKL = BKF$; conseq. $\angle GKL + LKB = 2$ right angles = $BKF + LKB$.

CASE II. When the points E and F are taken without the points A and B.

AE

$AE : AG :: AH : AB$, and $EB : LB :: HB : FB$, theref. in the Δ s
 AEH , EBH , which are respectively similar to AGB ,
 BLF . we have $\angle AGB = \angle AEH = \angle BLF$.

Also, $BH : BK :: BG : BL$, theref. the Δ BHG is sim. to BKL ,
 and $\angle BGH = \angle BLK$. Whence $\angle AGB + \angle BGH =$
 2 right angles $= \angle BLF = \angle BLK$.

Consequently the points L, K, F are in a straight line. *Q. E. D.*

The same, by Mr. James Cunliffe, Fig. 424.

Draw the lines GE , FK , and KL . Then by hypothesis,
 $AE \cdot AB = AC^2 = AG \cdot AH$, by Euc. 36, III. whence $AB :$
 $AG :: AH : AE$, therefore

$\angle AEG = \angle AHB$, by Euc. 6, VI; whence $\angle GEB = \angle GKL$
 $=$ supp. of $\angle H$. Again

by hyp. $BF \cdot BE = BD^2 = BK \cdot KG$, whence Euc. 16, VI. $BF :$
 $BG :: BK : BE$, therefore

Eu. 16, VI. $\angle FKB = \angle GEB = \angle GKL$, from what is done above,
 therefore the points K, L, F , are in the same straight line, Euc. 15, I.

Q. E. D.

The same, by Mr. John Harris. Fig. 426.

Join EK , EG , and FG . Because, by hypothesis,
 rect. $EAB = AC^2 = GAH$, the points E, G, H, B , are in a circle,
 theref. $\angle AEG = \angle GHB = \angle LKB$. Again, by the hypothesis,
 rect. $EBF = BD^2 = KBG$, therefore the points E, F, G, K , are
 in a circle, conseq.

$\angle AEG = \angle FKG$; hence $\angle FKG = \angle LKB$, and theref. LKF is a
 right line. *Q. E. D.*

The same, by Mr. John Lowry. Fig. 426.

Draw GE to meet the circle at Q , and join QL . Then,
 QL is parallel to AB , by Prop. XXXI. and by hyp. $EBF =$
 $BD^2 = BKG$;

therefore the points E, F, G, K are in a circle, hence

$\angle FKG = \angle FEQ = \angle EQL =$ (Eu. 22, III.) $\angle GHL = \angle LKB$,
 that is $\angle FKG = \angle LKB$;

therefore the points L, K, F , are in a right line. *Q. E. D.*

PROP. XXXVI. Fig. 427, 428, Pl. 23.

Demonstrated by Mr. Colin Campbell. Fig. 427.

About the Δ circumscribe the circle ABC , and produce AD
 to meet it again in a ; draw the chord $ac = AC$, and in ac , aA
 let af , ad be taken $=$ to AF , AD respectively, and join df , cd ,
 and CB intersecting aA in P .

Since

Since arc $ac = AC$ (Eu. 28, III.), arc $aC = Ac$, and therefore $\angle DAC = dac$ (27, III.); conseq. the $\Delta s ade, adf$, are equal and equiangular to ADC, ADF (Eu. 4, I.). Wherefore $\angle Pca = BAP$ (27, III.) $= ADF$ (29, I.) $= adf$; therefore the $\Delta s lPa, daf$ are equiangular, as also are BPA, AED , because $\angle AED = AFD$ (34, I.) $= adf$ (by constr.) $= aPc = BPA$ (15, I.) and $\angle BAP$ common. Whence

$aP : af (AF) :: ac (AC) : ad (AD)$, and $PA : EA :: BA : DA$ (4, VI.); therref.

$aP \cdot DA = FAC, PAD = EAB$ (Eu. 16, VI.); hence

$aP \cdot DA + PAD = DA \cdot (AD \cdot Da) = DA^2 + ADa = AD^2 + BDC = BAE + CAF.$ Q. E. D.

The same, by Mr. James Cunliffe. Fig. 427.

Describe a circle about the Δ by Eu. 5, IV. and produce AD to meet it at a and join Ba , also draw the line FG meeting AD and AB in H and G , so that the $\angle FGA$ may be $=$ the $\angle BCA$. Then the $\angle BaA = BCA$ (Eu. 21, III.) $= FGA$, by constr.; whence the $\Delta s AHG, ABa$ are similar, as also the $\Delta s AFG$ and ABC . Hence $\angle AHG = DHF$ (Eu. 15, I.) $= ABa$ by what is done above, also $\angle FDH = BAa$, (Eu. 27, I.), whence the $\Delta s DHF$ and ABa are similar. Therefore, by the sim. $\Delta s AFG$ and ABC ,

$AF : AG :: AB : AC$, whence, Euc. 16, VI. $AF \times AC = AG \times AB$.

Also because of the sim. $\Delta s AHG, ABa$, $AH : AG :: AB : Aa$, whence

$AH \times Aa = AG \times AB = AF \times AC$. Again, by the sim. $\Delta s DHF$ and ABa ,

$DH : DF :: AB : Aa$, whence $DH \times Aa = DF \times AB = AE \times AB$. Therefore

$AE \times AB + AF \times AC = DH \times Aa + AH \times Aa = AD \times Aa = AD \times (AD + Da) = AD^2 + AD \times Da = AD^2 + BD \times DC$,
Eu. 35, III. Q. E. D.

Mr. Harris refers to Art. II. VOL. I. REPOSITORY, for a demonstration.

The same, by Mr. Lowry. Fig. 428.

The truth of this proposition will easily appear from Prop. B. Art. 34, Vol. I. Repository. For if FI, EH , be drawn \parallel to BC , and FG, EQ to AD , it is evident that FI is $= EH$, (or $DG = DQ$) and $AI = DH$; hence the rect. $DAH + DAI = AD^2$. Again, by reason of the parallels, $EH (QD) : BD :: DC : BC$, or $QD \cdot BC =$ rect. BDC ; but the rect. $BDQ + CDG$ is $= QD \cdot BC$; therefore $BDQ + CDG = BDC$; hence by the Prop. above cited the rect. $BAE + CAF = AD^2 + BDG$. Q. E. D.

ARTICLE

ARTICLE XLIII.

Answers to the Mathematical Questions proposed in ARTICLE XXVIII. No. VIII.

I. QUESTION 157, answered by Mr. John Croudace, Student under Mr. Rutherford, Teacher of the Mathematics at Lancaster, Durham.

SINCE the \triangle is a right angle, the sum of the squares of the two sides = to the square of the hypotenuse, by Eu. 47, 1.

that is, $x^{10x} + x^{12x} = x^{14x}$; this equation divided by

x^{10x} gives $1 + x^{2x} = x^{4x}$, or $x^{4x} - x^{2x} = 1$; now

put $y = x^x$ and the equation will become $y^4 - y^2 = 1$,

hence by completing the square, &c. $y = \sqrt{\frac{1}{2}(\sqrt{5} + 1)}$. Con-

sequently $x^{5x} = y^5 = \left[\frac{1}{2}(\sqrt{5} + 1)\right]^{\frac{5}{2}} = 8.33019 =$ the base;

$x^{6x} = y^6 = \left[\frac{1}{2}(\sqrt{5} + 1)\right]^3 = 4.2366 =$ the perpendicular;

and $x^{7x} = y^7 = \left[\frac{1}{2}(\sqrt{5} + 1)\right]^{\frac{7}{2}} = 5.38836 =$ the hypotenuse; therefore the area is $= 7.0534549$.

The same answered by Mr. John Johnson, Birmingham.

By Eu. I. 47, $x^{10x} + x^{12x} = x^{14x}$, and by division and transposition, $x^{4x} - x^{2x} = 1$, and by completing the square,

&c. $x^{2x} = \frac{1}{2}(1 + \sqrt{5})$. Hence the base is $\frac{1}{2}(1 + \sqrt{5})^{\frac{1}{x}}$

$=$

$$= 3.330; \text{ perpendicular} = \frac{1}{2}(1 + \sqrt{5})^{\frac{1}{2}} = 4.236; \text{ hypo-} \\ \text{thenuse} = \frac{1}{2}(1 + \sqrt{5})^{\frac{7}{2}} = 5.388, \text{ and the area} = \frac{1}{2} \times \\ \frac{1}{2}(1 + \sqrt{5})^{\frac{11}{2}} = 7.05345.$$

The same answered by Mr. David Davis, Warminster School, Wilts.

By the property of right-angled Δ s, we have $x^{10x} + x^{12x} = x^{14x}$, or $x^{4x} - x^{2x} = 1$; putting $y + \frac{1}{2} = x^{2x}$, the equation will then become $y^2 - \frac{1}{4} = 1$, or $y^2 = 5 \div 4$, or $y = \sqrt{\frac{5}{4}}$; hence $x^{2x} = y + \frac{1}{2} = \frac{1}{2}\sqrt{\frac{5}{4}}$, or $x^x = \sqrt{\frac{1}{2} + \sqrt{\frac{1}{4}}}$ and conseq. $x^{5x} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{5}}^{\frac{5}{2}} = 3.330$, the base; $x^{6x} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{5}}^3 = 4.236$ the perpendicular; and $x^{7x} = \sqrt{\frac{1}{2} + \frac{1}{2}\sqrt{5}}^{\frac{7}{2}} = 5.388 = \text{hypotenuse}.$

Neat solutions to this question were also received from Messrs. Barron, Blackwell, Buffham, Bulmer, Burdon, Evans, Gregory, Harris, Lowry, May, Merones Minor, Surtees, Swale, and Tyro Philo Mathematicus.

II. QUESTION 158, answered by Miss Susan May.

Dr. Hutton hath shewn, at pa. 168, vol. i. of his Mathematical and Philosophical Dictionary, that $63551 \times \log. \text{ of } D \div d$ is a general expression for the altitude, in feet, when d is the density of the air at that height, and D the density at the earth's surface, the air being at a mean temperature, 55° . Hence we have $63551 \times \log. \text{ of } (1 \div \frac{1}{2}) = 63551 \times \log. \text{ of } 2 = 19130.757$ feet, the height required. On the same page the Dr. shews how to find the height at any other temperature; thus $1000 \times \log. \text{ of } D \div d$ is an expression for the height, in fathoms, when the temperature is at 31° . And to find it, at any other temperature, increase the height found for 31° , by the 435th part of that height for each degree above that temperature, and the sum will be the height required.

The

The same, answered by Mr. G. Buffham, Boston.

Let a be the altitude of the mountain, D the density at the earth's surface $= 1$, and d the density at the top of mountain $= \frac{1}{2}$. By Hutton's Course of Mathematics, page 235, Vol. II. $a = 63551 \times \log. D \div d = 63551 \times \log. 2 = 19130.757$ feet the height of the mountain, when the thermometer stands at 55° .

The same, answered by Mr. John Barron, Schoolmaster, Spilshy, Lincolnshire.

By page 236, vol. II. of Dr. Hutton's Course of Mathematics we have $1000 \times \log. (1 \div .5) = 3010.3$ fathoms for the height of the mountain, when the temperature of the air is at 31° .

The same, answered by Mr. Wm. Burdon, Acafter-Malbis, York.

Mr. Vince, at pr. 52, cor. 2, of his hydrostatics, says, "from experiments on the density of air at the bottom and top of hills, Mr. Cotes (Hydrostatics p. 103.) collected, that, at the altitude of 7 miles, the density was 4 times less than at the earth's surface, or $= \frac{1}{4}$." Hence if x = the dist. above the earth's surface, in miles, and r = the rarity of the air at that height, it will be as $\log. 4 : \log. r :: 7 : x = 1.1626 \times \log. r$ (in miles) $= 3\frac{1}{2}$ miles, the height of the mountain. Mr. Vince further remarks, that, "this rule supposes the density to be as the compressive force, which is not true, unless the temperature remains the same, but as the temperature is found to be very different, at the same time, at different altitudes, the rule will require a correction, according to the altitudes of the thermometers of the two places. Omitting, however, this correction, the density of the air, at the altitude of 45 miles, is found to be 7420 times less than at the earth's surface: and yet, from observations on the twilight, the rays of light are sensibly affected by the air at that altitude.

The same, answered by Tyro Philo Mathematicus, of Hull.

Let $r = 21008460$ feet = radius of the earth,
 $d = 1$ = density of the air, at the earth's surface,
 $h = 29725$ feet = height of an homogeneous atmosphere
 every where of the same density d ,
 $m = 2.302585$,
 x = the required altitude, taken from the earth's centre, and
 $D = \frac{1}{2}$ = density of the air, at that altitude. Then, allowing gravity to be inversely as the square of the distance from the
 centre,

centre, by *Emerson's Fluxions*, pa. 389, $D \div d =$ a number whose log. is $(r^2 - rx) \div mhx$; therefore, (putting the log. of $D \div d = -.30103 = -l$), $r^2 - rx = -mhlx$, and $x = r^2 \div (r - mhl)$; hence $x - r = mhlr \div (r - mhl) = 20624.026$ feet, the height of the mountain as required.

According to one or other of these methods the question was answered by Messrs. Blackwell, Bulmer, Croudace, Gregory, Harris, Johnson, Lowry, Merones Minor, and Thornoby.

III. QUESTION 159, answered by Miss May.

Let W = the weight of one square yard of the stuff composing the balloon,

w = the weight to be raised, $c = 3.14159$,

A = density of one solid yard of air, at the given height which may easily be found, by reversing the rule used in the last question,

a = density of the air with which the balloon is filled, and

x = the required diameter of the balloon. Then

cx^3 = the surface of the sphere, and $cx^3 \div 6$ = its solidity: hence

Wcx^2 = the weight of the balloon, and

$\frac{1}{6}acx^3$ = the weight of the air with which the balloon is filled; therefore

$\frac{1}{6}acx^3 + Wcx^2 + w = \frac{1}{6}Acx^3$, by the nature of the question. Hence

$$x^3 - \frac{6W}{A-a} x^2 = \frac{6w}{c(A-a)}, \text{ the THEOREM required.}$$

The same, answered by Mr. John Surtees, Bishop-Wearmouth, Sunderland.

Let x = the diameter of the balloon, h = the height to which it will rise,

W = the weight to be raised, w = the weight of one foot of the cover,

d = the weight of one cubic foot of air at the earth's surface,

a = the weight of one cubic foot of air with which the balloon is filled,

$c = 3.1416$, and r = radius of the earth. Then

πwx^2 = the weight of the balloon, and

$\frac{1}{6}acx^3$ = the weight of the contained air. And by *Emerson's Flux. pa. 389*, the weight of one cubic foot of air, at the height h , above

k , above the surface of the earth, is $\pm d \times$ number belonging to the log. of $(r \times - k) \div (68444 \times r + k) = (6cwx^2 + acx^3 + 6W) \div cx^3$, by the question. Hence x may be found.

The same, answered by Mr. Croudece, Lanchester.

Let x = the internal diam. of the balloon, y = its thickness,
 a = the given height it is to ascend to,
 b = the weight to be raised,
 c = the specific gravity of the air with which the balloon is filled,
 e = the weight of a yard of the substance composing it,
 n = .5236, and d = the specific gravity of common air. Then

$x + 2y$ = the external diameter,

$n(x + 2y)^3$ = the whole solidity.

nx^3 = the internal capacity,

cnx^3 = the weight of the air its filled with,

$dn(x + 2y)^3$ = weight of a mass of air of the same size as the balloon.

$n((x + 2y)^3 - x^3)$ = solidity of the part included between the two diameters, and

$en(x + 2y)^3 - x^3$ = its weight: Hence $en(x + 2y)^3 - x^3 + cnx^3 + b$ = the whole weight of the balloon and weight to be raised. But as the balloon will ascend and rest at such a height in the air where it will be of equal weight with the same bulk of it, and theref. where its density is to the density of the surface as

$en(x + 2y)^3 - x^3 + cnx^3 + b$ to $dn(x + 2y)^3$. Hence by

Emerson's Flux. 389th page, $68444 \times \log. \text{ of } dn \times x + 2y)^3 \div$

$(en \times (x + 2y)^3 - x^3 + cnx^3 + b) = a$. And taking y a known quantity, the theorem is manifest.

True solutions were also received from Messrs. Blackwell, Gregory Harris, Johnson, Lowry, Merones Minor, and Thornoby.

IV. QUESTION 160, answered by Mr. Burdon, Acafter-Malbis.

Fig. 429, Pl. 23. Let ABCD be the oblique cylinder, then if BD be \perp to AB, the pillar will just support itself by the principles of mechanics.—Now, by trigonometry, $\text{rad.} : AB :: \tan. \angle A : BT :: AB \sqrt{3}$, and by mensuration the solidity is $.7854 \sqrt{3} \times AB^3$
 VOL. II. 2 F

$= 26.548148$ cubic feet, by specific gravity: Whence $AB = \sqrt[3]{19.51564} = 2.6923$ feet the diam. of the base of the pillar and $BD = 4.6639$ feet its perpendicular altitude.

The same, answered by Miss May.

Fig. 430, Pl. 23. Let ABCD represent the cylinder in its inclined position; draw the diagonal BC, and from O the middle of BC draw OE \perp to CD. Then, it is evident, from the nature of the centre of gravity, that the cylinder will just support itself from falling when BC is \perp to the horizon CQ. Now, by experiments, it is found, that a cubic foot of marble weighs 2700 ounces, therefore, the content of the cylinder is $= 26.54814$ cubic feet, also by the quest. $\angle OCE = 60^\circ$, and $\angle COE = 30^\circ$; hence, by trigonom. $OE = CE \frac{1}{2}$, and by mens. $4CE^2 \times 2OE \times .7854 = (4 \div 3) \times OE^2 \times 2OE \times .7854 = (8 \div 3) \times OE^3 \times .7854 = 26.54814$. Whence $OE = \sqrt[3]{3 \times 26.54814 \div (8 \times .7854)} = 2.3314$. Conseq. the height is 4.6628 and the diam. of the base is 2.6932 feet.

The same, answered by Mr. Johnson, Birmingham.

Writers on mechanics have shewn, that the diagonal of the pillar, when it just supports itself from falling, will be \perp to the horizon. Hence, as 2700 (the sp. gr. of marble) : 1 :: 71680 ounces, or two tons : 26.54814814 feet, the solid content of the pillar. Again let x = the diam. of its base, a = sine of 60° , b = its cosine, then as $b : x :: a : ax \div b$ = the len. of the pillar, and by mens. $x^3 (ax \div b) \times .7854 = 26.54814814$; hence $x = 2.69$, and the length $= 4.659$ feet.

The same, answered by Mr. John Barron, Spilsby.

This Gentleman, after premising that CB, (fig. 430, pl. 23.) must be \perp to the horizon, puts a = content of the pillar, $d = .7854$, s and c = sine and cosine of $\angle ABC$, x = diam, and y = the height of the pillar. By hydrostatics, as 2700 : 761800 :: 1728 : 45875.2 the content in inches = a , and $dx^2y = a$, or $y = a \div dx^2$. Again, by trigonom. as $s : y :: c : x$ and $cy = sr$, or $y = sx \div c$. Hence, putting these two values of y equal to each other, we have $sx \div c = a \div dx^2$, or $x = \sqrt{ac \div ds} = 32.3079$ and $y = 55.9589573$, the dimensions of the pillar.

The

The same, answered by Mr. Croudace, Lancheffer.

Fig. 430, Pl. 23. Let ABCD represent the cylinder; draw $BC \perp$ to CQ and produce BD to Q . Then, as the sp. gr. of marble 2700 : 2 tons :: 1728 : 45875.2 inches = a , the solid content of the pillar. Put $CD = x$, $p = .7854$, $t = \tan. \angle DCQ = 30^\circ$. Hence $a \div px^2 = BD$, and $\text{rad.} : DC :: \tan. \angle DCQ : DQ = tx$, also $DQ : DC :: DC : BD = x \div t$, theref.
 $x \div t = a \div px^2$. This equation reduced gives $x = \sqrt[3]{ta \div p} = 32.30793 = DC$ and $BD = 55.95896$.

This question was also ingeniously answered by Messrs. Blackwell, Bulmer, Francis, Gregory, Harris, Lowry, Marrat, Merones Minor, Squire, and Tyro Philo Mathematicus.

V. QUESTION 161, answered by Mr. J. H. Swale.

CONSTRUCTION. Fig. 431, Pl. 23. In any line take $BG =$ the given dist. and produce it to A , so that the rect. $BG \cdot GA$ may be $=$ to the given area. Upon AB let a semicircle be described, and at B erect the \perp BD making the rect. $AB \cdot BD = 2AF \cdot FB$, and let DE be drawn \parallel to AB meeting the semicircle in C ; join AC , CB , and ACB will be the Δ required.

DEMONSTRATION. Make $AH = AG$, $BK = BG$, and at G , H , and K let \perp s be erected to meet each other at L the centre of the inscribed circle. Bisect AB in M and join CM . Then since BG is $=$ the given distance, we have only to prove, that the rectangle of the segments of the hypotenuse of any right angled Δ , made by the point of contact with the inscribed circle, is $=$ to the area. Now

$AG \cdot GB = (AM - MG) \cdot (AM + MG) = AM^2 - MG^2$; and
 $4AG \cdot GB = 4AM^2 - 4MG^2 = AB^2 - 4MG^2$: but
 $AB^2 = AC^2 + BC^2$, and, (because $2GM = AG - BG = AC - BC$),
 $4GM^2 = BC^2 + AC^2 - 2BC \cdot CA$. Therefore
 $2BC \cdot CA = BA^2 - 4GM^2 = 4AG \cdot GB$, and.
 $BC \cdot CA = 2AG \cdot GB = 2\Delta ACB$: Consequently
the rectangle $AG \cdot GB$ is $=$ to the ΔACB . Q. E. D.

COROLLARY. The area of any right angled plane triangle is equal to the difference between the square of half the hypotenuse and the square of half the difference of the legs.

The same, answered by Mr. Burdon, Acafter-Malbis.

After constructing the problem exactly like Mr. Swale, Mr. Burdon proves that the rect. $AG \cdot GB$ is $=$ to the ΔACB by the following

DEMONSTRATION. If L be the centre of the inscribed circle, and G, H, K the points of contact, then will $AG = AH$, $BG = BK$, and $HC = CK$; conseq.

$AC = AG + HC$, and $BC = BG + HC$, therefore

$(AG + HC) \cdot (BG + HC) = AC \cdot BC$, and, by Eu. I. 47,

$(AG + HC)^2 + (BG + HC)^2 = (AG + GB)^2$; subtract twice the former of these expressions from the latter, so shall $\frac{1}{2} AC \cdot BC = AG \cdot GB$, and by Euc. I. 41, the $\triangle ABC = \frac{1}{2}$ rect. $AD =$ the given area by construction.

Calculation. Dividing the given area of the \triangle by BG (6) gives $AG = 9$, therof. $AB = 15$. Again dividing the given area by AG gives $BD = CF = 7\frac{1}{2}$; but $MC = MB = \frac{1}{2} AB = 7\frac{1}{2}$; hence $MF = \sqrt{(MC - CF)^2} = 2\frac{1}{2}$, and $AF = 9\frac{1}{2}$. Conseq. $AC = \sqrt{(CF^2 + AF^2)} = 12$, and $BC = 9$ chains.

Mr. Andrews, of Cork, also gives the same construction, and thus proves that the rect. $AG \cdot GB$ is $=$ to the area of the $\triangle ACB$.

Complete the parallelogram $ACBN$ and through L , the centre of the inscribed circle, let QK and HT be drawn \parallel to AC and CB respectively.

Then, by the lem. on pa. 345, Simpson's Alg. the area of the $\triangle ACB$ is $=$ to the rect. $QC +$ the rect. KT ; but the rect. NC is $=$ to $2 \triangle ACB$, therof. the rect. $LN =$ the $\triangle ACB$. Moreover $AG = AH = QL = NT$, and $BG = BK = LT$; conseq. $AG \cdot GB =$ the given area ACB by construction.

The same, answered by Mr. Johnson, Birmingham.

ANALYSIS. Suppose the problem really solved, and lines drawn as in fig. 481, pl. 23, and that BG is $= 6$ chains the given distance. Then, at page 369, Vol. I. Prop. VIII, Art. 32, of the *Repository*, it is proved, that $AG \cdot GB = VM \cdot MW$, but $AM = MV$, therefore $AG \cdot GB = AM \cdot MW =$ the given area of the $\triangle ACB$. Now BG being given, AG will also be given, and conseq. the $\perp CF$ becomes known: Hence this

CONSTRUCTION. Take $BG =$ the given distance, and add thereto GA so that the rect. $AG \cdot GB =$ the area of the \triangle ; upon AB as a diam. describe a circle, bisect AB in M , and through M at right angles to AB draw VS to cut the circle in V and S , and in MS take MW so that the rect. $VM \cdot MW$ may be $=$ the given area; through W , \parallel to AB , draw WC to cut the circle in C and join AC, CB , so shall ACB be the \triangle required, as is evident from the Analysis.

The same, answered by Miss Susan May.

Suppose $\triangle ACB$, (fig. 431, pl. 23,) the \triangle required, L the centre of the inscr. circle, and G its point of contact with the hypotenuse; then, by the question, there are given BG and the area. About the \triangle let a circle be described, and draw CL to meet it at V , which will evidently be the middle of the arc AVB . Now, it is evident, from *Cor. 8, Prop. III. Art. II. Vol. II. of the Repository*, that the area of the $\triangle ACB$ is $=$ to the rect. $AG \cdot GB$, hence AG is given. Again from the Art. above cited, it is evident, that $VB = VL$: Hence this

CONSTRUCTION. Take $BG =$ the given segment and make the rectangle $BGA =$ the given area, on AB , as a diam. describe a circle and draw $GH \perp$ to AB , and from V the middle of the arc AB , to LG apply $VL = VB$ and draw VL to meet the circle at C , and join AC , CB , and it is done, as is evident from the Analysis.

Geometrical solutions were also received from Messrs. Cunliffe, Harris, Hill, and Lowry.

The same, answered by Mr. Gregory, the proposer.

Fig. 430, Pl. 23. Let ABC be the right $\angle d$ \triangle , GHK the inscribed circle, touching the sides of the \triangle in the points G , H , and K . Then $BG = BK$, $AG = AH$, $KC = CH = LH = LK$, L being the centre of the inscribed circle. Since BGL and BLF are $= \triangle s$, the sum of them, that is the quadrilateral $BGLF = BG \cdot GL$; for the like reason the quadrilateral $AHLG = AG \cdot GL$; and the square $LKCH$ is evidently $= GL \cdot GL$. The sum of these, or $(BG + GA + GL) \times GL$ is equal to the area of the $\triangle ABC$. But $BC = BG + GL$, and $AC = AG + GL$; therefore $(BG + GL) \cdot (AG + GL) = BG \cdot AG + AG \cdot GL + BG \cdot GL + GL \cdot GL$, $=$ twice the area of ABC , which is $=$ to $2BG \cdot GL + 2GA \cdot GL + 2GL \cdot GL$; as appears from the foregoing. From each of these equals take away $BG \cdot GL + AG \cdot GL + GL \cdot GL$, and there remains $BG \cdot AG = BG \cdot GL + AG \cdot GL + GL \cdot GL$; but $(BG + AG + GL) \cdot GL$ is $=$ to the area of the $\triangle ABC$, therefore $BG \cdot AG$ is $=$ to the area of the said $\triangle ABC$. Now, in the question, we have given $BG = 8$ chains, and the area of the $\triangle = 51.1 : 94 = 54$ sq chains; whence from the above theorem $54 \div 6 = 9 = AG$, and hence $6 + 9 = 15 = AB$. Put $HC = CK = x$, then $AC = AG + HC = x + 9$ and $BC = BG + KC = x + 6$. Wherefore $(x + 6)(x + 9) = 2\triangle ABC$; that is, $x^2 + 15x + 54 = 108$; from which equation x is easily found $= 3 = HC = CK$; and

conseq. $AC = 9 + 9 = 18$ ch, and $BC = 9 + 6 = 9$ ch, the dimensions required.

Or, since, $(AB + GL) \cdot GL = \text{area of } ABC$, as may be easily proved, if x represent LG or HC as above, we shall have $(15 + x)x$, or $15x + x^2 = 54$, which completed and reduced gives $x = 3$ chains, as before determined.

The same, by Mr. Bulmer, Sunderland.

Let ACB be the Δ , and G, H, K the points of contact with the infer. circle; then we have given $BG = BK = 6$ and the area $54 = a$.

Put the rad. of the infer. circle $= x$, and $AG = AH = y$, then $AB = y + 6$, $BC = x + 6$, and $AC = x + y$. Now, by Eu. I. 47,

$(y+6)^2 = (x+6)^2 + (x+y)^2$, also by the property of the figure, $(x+6) \cdot (x+y) = 2a$; these equations reduced produce

$$x^2 + 12y + 36 = x^2 + 12x + 36 + 2xy + y^2, \text{ or } 2x^2 + 2xy + 12x = 12y, \text{ and}$$

$$x^2 + xy + 6x + 6y = 2a. \text{ Multiply this equation by 2, the product is } 2x^2 + 2xy + 12x + 12y = 4a; \text{ from this equation take}$$

$$2x^2 + 2xy + 12x = 12y, \text{ and there remains } 12y = 4a - 12y, \text{ or } 24y = 4a, \text{ or } 6y = a, \therefore y = a \div 6 = 9.$$

Again, by the property of the figure, $(6+x)^2 + (9+x)^2 = 15^2$, or $2x^2 + 30x = 108$, or $x^2 + 15x = 54$; hence $x = 3$, and the three sides of the Δ are 9, 12, and 15 chains.

The same, by Mr. T. Squire, Baldock.

Let ACB represent the triangular field, the area of which is given $= 54$ sq. chains, which put $= a$, and let L be the cen. of the infer. circle, and G, H, K the points where the circle touches the sides of the Δ . Put $BG = b = 6$ ch. the given diff. and the rad. of the infer. circle $= x$, then $BC = b + x$, and by trigonometry, $b : 1 :: x : a \div b = \tan. \angle GBL$, or LBK , for the line BL bisects the $\angle B$. By pa. 34, *Crackell's Translation of Maudslayi's Trigonometry*, the tan. of the double arc is $= 2bx \div (b^2 - x^2) = \tan. \angle ABC$; again, as $1 : 2bx \div (b^2 - x^2) :: b + x : 2bx \div (b + x) \div (b - x) (b + x) = 2bx \div (b - x) = AC$, and $AC \cdot BC = 2bx (b + x) \div (b - x) = 2a$; hence $x = 3$, and the three sides of the field are 9, 12, and 15 chains respectively.

Ingénieux algebraical solutions were also received from Messrs. Barron, Blackwell, Buffham, Croudace, Francis, Marjat, Merona Minor, Thornaby, and Tyro Philo Mathematicus.

VI. QUESTION 162, answered by Mr. W. Francis, Teacher
of the Mathematics, at Hamstead, Middlesex.

The proposer of this question seems to have been mistaken, in the composition of his globe, for, it is evident, it could never have produced the effect he describes had it been made of glass or any other substance more heavy than the liquid in which it was immersed. Let us suppose he wished only to know the content of that segment of the globe which would be forced under the water by the weight of the globe only. Then since the solidity of a globe one foot in diameter is 904.7808 cubic inches, and that of one 11.9 inches in diameter is 882.3492524, it follows that 22.4315 inches is the content of the glass. And as 1000 : 2600 :: 22.4315 : 58.3219 inches the content of the water displaced by it. Now put x = the depth of the segment immersed, then $(12 \times 3 - 2x) \times x^2 \times .5236 = 58.3219$ by mensuration, that is, $18x^3 - x^3 = 55.69318$, whence $x = 1.8574$. It now remains to find the content of the segment within the globe even with the water's surface, or 1.8074 in depth, the diameter being 11.9 inches. Whence by mensuration $(11.9 \times 3 - 2 \times 1.8074) \times 1.8074^2 \times .5236 = 54.8798$ solid inches the quantity sought.

The same, by Mr. Olinthus Gregory, Bookseller, Cambridge.

The ingenious proposer of this question has inadvertently fallen into an error in the enunciation of it: for since glass is heavier than water, it is impossible, according to the principles of hydrostatics, that the water in a glass sphere should be on a level with the water in which it is placed, unless some extraneous power be applied to sustain the sphere in that situation. In order, therefore, to render the question capable of a solution, conformably to the proposer's views, either the liquor in the sphere must be of a less specific gravity than water, or the shell must be composed of some substance which is lighter than water. First, let us suppose the liquor to be of a less specific gravity than water, and let the specific gravities of glass and the liquor be r and s respectively, that of water being unity. Knowing the diam. and thickness of the glass sphere its specific gravity r being given, the relative weight may be readily found, this we will denote by g . Then, if z be put for the altitude of the liquor in the sphere, a for the axis of the sphere, t for the thickness of the glass, and c for .5236, we shall have $c(3a(z+t)^3 - 2(z+t)^3)$ to denote the relative weight of the water displaced, and $cr(3az^3 - 6tz^3 - 2z^3)$ for that of the liquor in the sphere. Hence we deduce the following equation:

$$c(3a(z+t)^3 - 2(z+t)^3) = cr(3az^3 - 6tz^3 - 2z^3) + g.$$

This

This may be reduced to a final equation of the third degree, from which, when s and t are properly chosen, the value of x may be ascertained. Or, agreeably to the second supposition, let us imagine the shell of the sphere to be composed of elm, the specific gravity of which is 600, that of water being 1000, and let us suppose the thickness of the shell instead of $\frac{1}{20}$ to be $\frac{1}{2}$ an inch. Then, it is evident, that the sphere will rest (as per quest.) when the weight of a portion of water, equal to the spherical segment immersed, is equal to the weight of the water in the sphere, added to the weight of the wood of which the shell is made; and, of course, the solid content of the elm which is immersed, is to that of the whole spherical shell, as the specific gravity of elm to that of water, or as 6 to 10. But the thickness of the shell being small, when compared with the axis of the sphere, we may, for an approximation, suppose the content of the part of the shell immersed in the water, to be as the surface immersed, or proportional to the altitude of the segment: and then, 11 inches being the internal axis of the hollow sphere, we shall have $11 \times .6 = 6.6$ inches, for the altitude of the water therein. This result will, on a comparison with more accurate methods, be found to vary not more than a thousandth part from the truth. It must be manifest that in these remarks the effects of air, either within or upon the sphere, have not been taken into consideration.

The same, answered by Tyro-Philo Mathematicus.

The enunciation of this question seems to contradict the principles of hydrostatics: but if instead of the words, "till the surface of the water in the sphere was on a level with the surface of the water in the vessel," be inserted, "till the surface of the water in the sphere was n inches below the surface of the water in the vessel," or if instead of water some other liquid, whose specific gravity is given less than that of water, had been contained in the sphere; then the quantity, in either case, may be determined as follows:

Let d = diameter of the sphere, (out side dimensions)
 t = thickness of the glass, p = 5936,
 $a, b, c,$ = the sp. gr. of glass, water, and the liquid contained in the sphere,
 n = the diff. of the altitudes of the liquid in the sphere, and the water in the vessel,
 and x = the altitude of the segment, or depth of the liquid in the sphere.

Then we shall have

$$p(d^3 - (d-t)^3) = (c-a)x^2 + (c-b)x^3$$

$p(a^3 - d - 2t)^3 \cdot a$ = the whole weight of the glass,

$p(3dx^3 - 6tx^2 - 2x^3) \cdot c$ = the whole weight of the liquid contained in the vessel, also

$p(3d \times x + n + t)^3 - 2 \times x + n + t)^3 \cdot b$ = the whole weight of the water displaced by the sphere, which must be = to the sum of the two expressions above, by hydrostatics. Hence we get this cubic equation (putting $n + t = r$).

$$\left. \begin{array}{l} + 2c \} x^3 + \frac{3db}{6br} \} \\ - 2b \} \quad + 6ct \} x^2 - \frac{6dbr}{6br^3} \} x + \frac{3dbr^3}{2br^3} \} \\ \quad - 3cd \} \quad - 3at^3 \} \\ \quad \quad \quad - 6adt^2 \} \end{array} \right\} = 0,$$

wherein the value of r must always be taken greater than t , when c is equal to a greater than b .

N. B. No notice is here taken of either internal or external air.

Similar observations were also received from Messrs. Marriat, the proposer, and Merones Minor.

VII. QUESTION 163, answered by Rev. L. Evans.

If we turn to Prop. XXII. Book II, of the Principia, we shall have, by the property of the hyperbola (vid. the figure to prop. just mentioned) $SB : SA :: Aa : Bb = SA \times Aa \div SB = SA^2 \div SB$, because SA and Aa are equal. In like manner $Ef = SA^2 \div SF$. Therefore $Aa - Bb = (SA \times SB - SA^2) \div SB$, and $Aa - Ef = (SA \times SF - SA^2) \div SF$. Consequently, $Aa - Ef : Aa - Bb = (SA \times SF - SA^2) \div SF : (SA \times SB - SA^2) \div SB = (SF - SA) \div SF : (SB - SA) \div SB = AF \div SF : AB \div SB = AF \times SB : AB \times SF$, as it ought to have been printed in the question.

The same, answered by Mr. Geo. Saunderson, London.

Correcting, in the question, $AF \times SB$ to $AF \times SA$, the solution will be thus:

By the property of the hyperbola $SA : SF :: Ef : Aa$; and $SA : SB :: Bb : Aa$. Whence (by inver. and conver.) $SF : AF :: Aa : Aa - Ef$; and $SB : AB :: Aa : Aa - Bb$; or $AB : SB :: Aa - Bb : Aa$; this compounded with the third,
 $SF \times$

$SF \times AB : AF \times SB :: Aa - Bb : Aa - Ff$; or as 1 :

$\frac{AF \times SB}{SF} = 1005325$, because $AB = 1$, which is a more

simple expression than 1 : $\frac{Aa - Ff}{Aa - Bb}$ its equal.

VIII. QUESTION 164, answered by Tyro Philo Mathematicus.

As 81 : 70 :: rad. : tang. of $40^\circ 50'$, the apparent altitude of the sun's upper limb, and applying the semidiam.— $16' 4''$, refraction— $1' 3''$, and parallax. $+ 7''$, the sun's true central altitude will be $= 40^\circ 33''$; now, in the spherical $\triangle ZPS$ (fig. 432, pl. 23.), let P be the pole, Z the zenith and S the sun's place. Then there are given ZS the co-alt. $= 49^\circ 27'$, $\angle PZS$ the azimuth $= 87^\circ$, and $\angle ZPS$ the hour angle $= 54^\circ$, to find PS and ZP thus. As sine $\angle ZPS$: sine ZS :: sine $\angle PZS$: sine PS $= 69^\circ 42' 20'' =$ co-declination, and as sine $\frac{1}{2} (PZS \oslash ZPS)$: sine $\frac{1}{2} (PZS + ZPS)$:: tan. $\frac{1}{2} (PS \oslash ZS)$: tan. $\frac{1}{2} ZP = 30^\circ 39' 43''$, and ZP $= 61^\circ 19' 26'' =$ co-altitude; hence the day was the 21st of May, and the latitude $28^\circ 40' 34''$ N.

The same, answered by Mr. T. Squire, Baldock.

By trigonom. as 70 : 81 :: rad. : $1.1571428 =$ the cotang. of $40^\circ 50'$, the sun's apparent altitude, but allowing for semidiameter and refraction, we have the sun's true alt. $= 40^\circ 32' 51''$; then in the spherical $\triangle ZPS$ (fig. 432. pl. 23.) we have given ZPS $= \angle$ of time, ZS $=$ co-alt. and $\angle PZS =$ azimuth angle, to find PS $= 69^\circ, 42' \frac{1}{3} =$ co-dec. and ZP $= 61^\circ 19' 26''$ the co-latitude.

The same, answered by Mr. J. Blackwell, Hungerford,

First, as $\sqrt{70^2 + 81^2}$: rad. :: 70 : sine of $40^\circ 50'$ the alt. of the sun's upper limb, and by allowing for semidiam. and refrac. we have the true alt. of the sun's centre $= 40^\circ 30' 57'' \frac{1}{2}$. Hence in the oblique angled sph. $\triangle SZP$ (fig. 432, pl. 23.) there are given the co-alt. ZS $= 49^\circ 27' 02'' \frac{1}{2}$, the $\angle SZP = 87^\circ$, and the $\angle SPZ = 54^\circ$ the meridian distance, to find ZP $= 61^\circ 19' 36''$ the co-lat. and SP $= 69^\circ, 42', 26''$. Whence the latitude was $28^\circ 40' 24''$ North, and the sun's declination answers to the 21st of May.

Solutions

Solutions to this question were also received from Messrs. Barron, Burdon, Croudace, Gregory, Johnson, Lowry, Mairat, and Merones Minor.

IX. QUESTION 165, answered by the proposer, Mr. George Brown, Teacher of the Mathematics, at Newcastle-upon-Tyne.

Let x and y = the sine and cosine of the required latitude,
 z and v = the sine and cosine of the required declination;
 then per spherics

$1 : x \div y :: z \div v : a$, that is, $xz = avy$, a being = sine of 30° the ascensional diff. and

$1 : x :: z : b$, that is, $ax = b$ = sine of $15^\circ 35'$ the altitude at fix, therefore,

$vy = b \div a$, by equating the two values of xz . Hence

$vy - xz = \frac{(1 \div a) - 1}{\text{the sum of the lat. and dec. and}} \times b = .2686396 = \cos. \text{ of } 70^\circ 25'$

$vy + xz = \frac{(1 \div a) + 1}{\text{their difference; therefore}} \times b = .8059188 = \cos. \text{ of } 36^\circ 18' 4''$

$\frac{1}{2} (74^\circ 25' + 36^\circ 18' 4'') = 55^\circ 21' 32''$, the latitude, North, and
 $\frac{1}{2} (74^\circ 25' - 36^\circ 18' 4'') = 19^\circ 3' 28''$, the declination answering to May 16.

The same, answered, by Merones Minor.

Put a = nat. cosine of $15^\circ 35'$, x = nat. cosine of (co-declination — co-latitude), and y = the nat. cosine of their sum, to radius 1. Then, by Emerson's Trigonomet. Cor. I. Prop. 42, B. III, we have

$x - y : x - a :: 2 : 1$, the versed sine of $90^\circ = 6$ hours, and

$x - y : x :: 2 : 1.5$ the versed sine of $120^\circ = 8$ hours :: $4 : 3$.

Hence $x = 3a = .8059188$ the cosine of $36^\circ 18'$, and $y = -a$ the cosine of $105^\circ 35'$. Wherefore the latitude was $55^\circ 21' \frac{1}{2}$ north, and the declination $19^\circ 3' \frac{1}{2}$ N, answering to the 16th of May.

The same otherwise, by Mr. Burdon, Acafter Malbis, York.

By spherics,

rad. : tan lat. :: tan. dec : sine of \angle of time from sun rise to fix, and
 rad. : sin lat. :: sin. dec : sine of the altitude at fix, therefore

$$1 : \frac{\tan \text{ lat.}}{\sin \text{ lat.}} :: \frac{\tan. \text{ dec.}}{\sin \text{ dec.}} : \frac{\sin. \text{ of } \angle \text{ of time}}{\sin. \text{ of alt. at fix}}, \text{ or}$$

1 : sec.

1 : sec. lat. :: sec. dec. : $\frac{\sin. \text{ of } \angle \text{ of time}}{\sin. \text{ alt. at fix}}$. Hence here

are given the rectangle of the tangents of two arcs and the rectangle of their secants to the same given radius to find those arcs, of which this may be the

CONSTRUCTION. Fig. 433, Pl. 23. Take CD = the given radius and produce it to E so that $CD \cdot DE$ may be = to the rect. of the tangents; bisect CE in F with the perpend. FG , and from E , to FG , apply EG = a 3rd proportional to $2CD$ and the side of the square denoting the rectangle of the secants. Then with the centre G and radius GE or GC describe a circle cutting AB (drawn \perp to CD) in A and B , so shall AD , DB , be the tangents of the required arcs.

For $AD \cdot DB = CD \cdot DE$ = the given rect. of the tangents, and $AC \cdot CB = 2EG \cdot CD$ = the given rect. of the secants, by constr.

The Calculation easily follows from the construction, giving BD = tangent of $19^\circ 3\frac{1}{2}'$ the sun's declination, answering to the 16th of May, and AD = tangent of $55^\circ 21\frac{1}{2}'$ the latitude of the place of observation.

The same again, by Tyro Philo Mathematicus, of Hull.

This quest. may be solved by the orthographic projection of the sphere, thus, let the circle HZR (fig. 434, pl. 23.) be the meridian, the arc Hm the sun's meridian altitude, its sine bm ; the arc Rn the midnight depression, its sine cn ; EQ the equator, mn a parallel of declination; Pp , at right angles to EQ , the six o'clock line; g the sun's place at rising; and sr the sink of his altitude at fix. Then by the principles of this projection, it will be as

Of: fQ ($:: rg : gn$) $:: sr : cn$, or, as

$\sin 30^\circ$: radius — $\sin 30^\circ :: \sin 15^\circ 35'$: $\sin 15^\circ 35' = Rn$, and as

Of: fE ($:: rg : gm$) $:: sr : bm$, or as

$\sin 30^\circ$: radius + $\sin 30^\circ :: \sin 15^\circ 35'$: $\sin 53^\circ 42' = Hm$, theref.

$HE = \frac{1}{2} (Hm + Rn) = 34^\circ 38\frac{1}{2}'$ = the colatitude, and

$Em = \frac{1}{2} (Hm - Rn) = 19^\circ 8\frac{1}{2}'$ = the declination. Hence the latitude $55^\circ 21\frac{1}{2}' N$, and the day the 16th of May.

Mr. Croudace likewise answered this Question.

X. QUESTION 166, answered by Merones Minor.

Let $ABRS$ be a section of the frustrum, $fHfH$ that of the sphere, and produce AR , BS to meet at D and draw the right lines as in figure 435, plate 23.

Then

Then by similar triangles,
 $AC-RT : TC :: RS : DT = 20 \div 3$; hence $DC = 80 \div 3$, and

$DS = \sqrt{(20 \div 3)^2 + 2^2} = (2 \div 3) \sqrt{109}$. Then
 $RS : AB :: DS : DB = (8 \div 3) \sqrt{109}$, and
 $CB : CD :: EF : DF = 20$, $DC : DB :: DF : DE = 2 \sqrt{109}$,
 $DB : DC :: DF : DG = 200 \div \sqrt{109}$. Hence
 $HD = DE - HE = 2 \sqrt{109} - 6$,
 $GH = GD - HD = 6 - 18 \div \sqrt{109}$, and
 $IH = (3 \sqrt{109} - 9) \div 5$.

Now, it is well known, (see quest. 225, of the Scientific Recepticle) that the cone IGK is = to the ungula IfHFK included between the sphere and cone, and that the cone Gab is = to the ungula fLrOeF. Now if MN be the surface of the water, before the globe be immersed, we obtain the content of the cone DMN = $p \times (1250 \div 9)$, p being = 3.14159 , and the sum of the contents of the cones DIK, IGK is = $p \times (2832 \div \sqrt{109} - 144)$, the latter of these taken from the former leaves $p \times (2546 \div 9 - 2832 \div \sqrt{109})$ for the quantity of water above the points of contact F, f, which must be = to the cone Gab. Therefore draw ikh parallel to MN touching the sphere at k and join Gi, Gk, also draw LaPbO for the surface of the water after the immersion of the sphere. Now, $Dh = DE + Eh = 2 \sqrt{109} + 6$, $Gh = Dh - DG = 6 + 18 \div \sqrt{109}$, and $ik = (6 \sqrt{109} + 18) \div 5$; then $ik^2 \times p \times \frac{1}{3} Gh = p \times (18144 + 2448 \sqrt{109}) \div 25 \sqrt{109} =$ the cone Gik. But, cone Gik : cone Gab :: $Gh^3 : GP^3$, or $\sqrt[3]{Gik} : \sqrt[3]{Gab} :: Gh : GP = 3.17533$. Whence $PQ = DG + GP = DQ = 5.6652$ inches, the height the water will rise as required.

The same, answered by Mr. John Blackwell, Hungerford.

Fig. 435, Pl. 23. Let ABRS represent the given vessel, or conical frustum. Then by sim. Δs , $\frac{1}{2} (AB - RS) = Bq : qS ::$

$BC : CD = 26^2$, and $\sqrt{DC^2 + BC^2} = BD = 27.8408$; again,

by sim. Δs , $BC : \sqrt{DC^2 + BC^2} :: EF (= 6) : ED = 20.8806$, and $ED + EF = Dh = 26.8806$; then $DC : AB :: Dh : ik = 16.12836$. Put $x = hP$ the height of the segment above the surface LO, then $dh : ik :: Dh - hP : LO$, and

$\sqrt{(EF^2 \text{ or } Eh^2 - Eh - hP)^2 \text{ or } PE^2} = Pe$, and, by mensuration,

$3 (Eh^2 - Eh - hP)^2 + hP^3 \times .5236 =$ solidity of the segment *rhe*. Now, by the nature of the question, the frustum LS

$$\left(3 \times \frac{dx \times ac}{a} + 2c \times \frac{dx + ac}{a} + c^2 \right) \times \frac{x}{4} \times \frac{x}{3} \times ps =$$

BN \times weight of GE. By the nature of the question, MI \times weight of DG + AI \times W = BN \times weight of GE + BI \times w, that is,

$$\left(3b^2 + 2bx \times \frac{dx + ac}{a} + \frac{dx + ac}{a}^2 \right) \times \frac{a-x}{4} \times \frac{a-x}{3} \times ps$$

$$+ (a-x). W \text{ is equal to } \left(3 + \frac{dx + ac}{a} + 2c \times \frac{dx + ac}{a} + c^2 \right)$$

$$\times \frac{x}{4} \times \frac{x}{3} \times ps + xw.$$

Hence $3a^2b^2 \times (a^2 - 2ax + x^2) + 2ab \times (a^2 - 2ax + x^2) \times (dx + ac) + (a^2 - 2ax + x^2) \times (d^2x^2 - 2acdx + a^2c^2) - 3x^2 \times (d^2x^2 - 2acdx + a^2c^2) - 2acx^2 \times (dx + ac) - a^2c^2x^2 + \frac{12a^2}{ps} \times (a-x). W - \frac{12a^2xw}{ps} = 0$. Whence

$$+ 3a^4b^2 - 6a^3b^2x + 3a^2b^2x^2 + 2abdx^2 - 2d^2x^4 = 0.$$

$$+ 2a^4bc + 2a^3bdx + 4a^2b^2dx^2 - 2ad^2x^3$$

$$+ a^4c^2 - 4a^3bcx + 2a^2bcx^2 - 7acdx^2$$

$$\frac{12a^3W}{ps} + 2acdx + a^2d^2x^2$$

$$- 2ac^2x - 2a^2edx^2$$

$$- \frac{12a^2W}{ps}x + 5a^2c^2x^2$$

$$- \frac{12a^2w}{ps}x$$

from which the value of x may be found.

XII. QUESTION 168. Several of the Editor's correspondents inform him, that they have attempted solutions to this question, but with hopeless success. The major part of them are induced to conclude, that some error has crept into the question, which renders a satisfactory answer impossible. As the proposer himself has not succeeded so well as might be desired, the Editor takes the liberty of requesting, he will reconsider the matter, and favour him with an accurate solution or the necessary corrections, for appearance in a future number.

XIII. QUESTION 169, *answered by Mr. J. Blackwell, the proposer.*

The latitude, day, and hour being given, the sun's alt. is found: $= 41^{\circ} 31' 2'' 7$, this $+ 1' 1''$ (refract.) $- 7''$ (paral) $= 41^{\circ} 32' 16'' 7$ the sun's true alt, the sine and cosine of which put $= s$ and c , to rad. 1, also put $a = 172.8$, $b = .78539$. And also let $3x$ and $2x$ denote the transverse and conjugate axes of the picture frame. Then by trigonom. $s : 2x :: c : 2cx \div s =$ the conjugate axis of the included space on the plane of the horizon. Theref. $(2cx \div s) \cdot 3x \cdot b = a$, by the quest, or $6bcx = as$, or $x = \sqrt{ac \div 6bc} = 5.699615$, theref. the axes of the frame are 17.098845 and 11.39923 . Whence its area is 153.0849 square inches.

The same, answered by Mr. Gregory, Bookseller and Teacher of the Mathematics, at Cambridge.

This question is not expressed so clearly as might be wished: but I suppose the proposer means that the horizontal plane, on which the shadow was measured, touched the lower part of the conjugate axis of the ellipsis—for, on any other supposition there are not sufficient data. Now, the latitude, declination, and hour angle being given, the alt. of the sun's upper limb is found by spherics, after proper correction for refraction, &c. to be $41^{\circ} 47' 50''$; and as the shadow of the ellipsis will be the greatest, under the circumstances allowed by the question, when its transverse axis is perpendicular to the solar rays, we shall have one axis, of the shadow (which it is well known will be either an ellipsis or a circle) equal to the transverse axis of the elliptical frame, and the other axis of the shadow in proportion to the conjugate of the frame, as the cotangent of the sun's apparent alt. to radius. Therefore, putting $3z$ and $2z$ for the transverse and conjugate of the frame, c for $.785398$ &c. and t for the cotang. of the alt. to rad. 1, we

have $c \times 3z \times 2zt = 6ctz = 172.8$ the area of the shadow, whence we get $z = \sqrt{(172.8) \div 6ct} = 5.725639$, and the axes of the frame are 17.176917 and 11.451278 respectively.

But in the question the area alone is required, and this will be represented by $3z \times 2z \times c$, or $6cz^2$: therof. as $6ctz^2 : 6cz^2 :: 172.8 : 172.8 \div t = 172.8 \div 1.1185482 = 154.48599$ square inches, area of the frame.

The same, answered by Mr. Wm. Francis.

Supposing the lower end of the conjugate diameter to touch the horizontal plane, the solution will be thus:

Having found, from the data, the apparent alt. of the sun's upper limb $= 41^\circ 50' 39''$, and since the conjugate and transverse axes are as 2 to 3; then as $41^\circ 50' 39'' : 2 :: 48^\circ 9' 19'' : 2.235$. Therefore the axes of the shadow are in proportion to each other as 3 to 2.234. Let $3x =$ the transverse, then $2.234x =$ the conjugate. And $3x \times 2.234x \times .7854 = 172.8$ by the question, that is, $5.2637508x^2 = 172.8$, or $x = 5.7296$. Hence as $3 : 3 \times 5.7296 :: 2.234 : 12.8$ the conjugate axis of the shadow. Again as $2.234 : 2 :: 12.8 : 11.4592$ the conjugate axis of the frame also $2 : 11.4592 :: 3 : 17.188797$ the transverse axis. Hence the area of the frame $= 154.700128$ square inches.

Ingenious solutions were also received from Messrs. Croudaec, Evans, Lowry, May, Merones Minor, and Thornoby.

XIV. QUESTION 170, answered by Mr. John Lowry.

CASE I. For the greatest ellipsis that can be inscribed in the parabola. Fig. 437, Pl. 23.

Suppose the thing done, and that DNCKEL is the greatest ellipsis that can be inscribed in the given parabola HCPEG, having one of its axes (NL) parallel, and the other (DK) perpendicular to the base of the parabola. Through the points of contact C and E, draw the tangents BCA, and BEF to intersect in B, and meet HG produced both ways at A and F; join AE, FC, and CE, and draw KR to touch the ellipsis at K, the extremity of the axis DK. Then, by Simpson on the *Max. et Min.* the ellipsis DCE is the greatest that can be inscribed in the $\triangle ABF$, when the sides AB, BF, AF, are bisected at the points of contact C, E, D; much more, then, will it be the greatest that can be inscribed in the parabola HCPEG, which is wholly included within the \triangle , therefore we have only to draw the tangent AB, or BF, so as to be bisected at the point of contact C, or E.

Now, by the property of the parabola $IP = BP$, and since $BE = EF$, by hypothesis, BI will be $= ID$, therof. $gPB = PD$,
and

and theref. $BD = 16 \div 3$. Again, ſince the ſides of the \triangle are biſected by the lines AE , FC , DB , the point of their interſection, O , will be the centre of the ellipſis, and by the property of the \triangle , $2OD = BO$, theref. $OD = BD \div 3 = 16 \div 9$, and $DK = 2OD = 32 \div 9 =$ one of the axes.

Again, by the property of the parabola $PD : PI :: DG^2 : IE^2 = 5^2 \div (4 \times 3)$, theref. $DF = 5 \div \sqrt{3}$, and $KR = 5 \div 3\sqrt{3}$: but by the property of the ellipſis $DF \cdot KR = OL^2 = 5^2 \div 3^2$, theref. $NL = 10 \div 3$ the other axis of the ellipſis. Wherefore the area is $= (320 \div 27) \times .7854$.

Note. When the $\triangle ABF$ is equilateral, the ellipſis becomes a circle, and BD is then $= DG\sqrt{3}$, $DP = \frac{2}{3} BD = \frac{2}{3} DG\sqrt{3}$, or $DP : HG :: \sqrt{3} : 8 \div 3$. Hence it follows, that if in any parabola, the ratio of the baſe HG to the abſciſſa DP , be greater than $8 \div 3 : \sqrt{3}$, the greater axis of the ellipſis will be \parallel to HG , but if in a leſs ratio the leſs axis will be \parallel to HG .

CASE II. For the greateſt parabola that can be inſcribed in the given ellipſis. Fig. 438, Pl. 23.

Suppoſe that BAC is the greateſt parabola that can be inſcribed in the given ellipſis $BACE$. Through the points of contact A , B , C , draw the tangents GAF , GBH , FCH , the point A being the extremity of one of the axes of the ellipſis, and the point of contact of the ellipſis and parabola. Then it is evident that the parabola BAC will be a *max*, when the rect. $AD \cdot BC$, to which it has a conſtant ratio, is a *max*; and by Simpson on the *max. et min.* that rectangle will be the greateſt when the tangents are biſected at the points A , B , C .

Now, if BF be drawn to interſect AE at I , I will be the centre of the ellipſis, theref. $AI = IE = EH$, and $AD = DH = IE + DE = 3DE = \frac{2}{3} AE$. But by the property of the ellipſis, if C be put to repreſent the conjugate axis, we have the ſquare of the tranſverſe $AE : C^2 :: AD \cdot DE = \frac{2}{3} AE^2 : BC^2 = \frac{2}{3} C^2$, theref. $BC = \frac{1}{3} C\sqrt{3}$, and therefore the area of the parabola is $= \frac{2}{3} AD \times \frac{1}{3} C\sqrt{3} = \frac{1}{3} AE \times C\sqrt{3} = 5\sqrt{3}$.

It is evident, that we ſhall obtain the ſame expreſſion for the area, if the point of contact A be at the extremity of the leſs axis of the ellipſis, ſo that the baſe of the parabola may be parallel to either of the axes of the ellipſis, the area in each caſe being the ſame.

The ſame, answered by Merones Minor.

Fig. 439, Pl. 23. Let BMN be the ſemiparabola and $DMAE$ the ſemiellipſis inſcribed therein: through the point of contact M draw the tangent TMt , and the ordinate PM , and alſo draw $MO \perp$ to Tt . Put $a = EB$, $p = 25 \div 16$ the param. of the parabola

bola and $x = BP$. Then, by *Emerson's Conics*, III. 3, $OP = \frac{1}{2}p$, also *ibid* 17, I. Cor. 2, $EP:ET::PD:DT$, and, by composition, $EP + ET:PD + DT::EP:PD$, that is, $2a:2x::a-x:x(a-x) \div a = PD$; then $CD = \frac{1}{2}(EP + PD) = (a^2 - x^2) \div 2a$, $PC = CD - PD = (a - x)^2 \div 2a$; also, *ibid*. 15, I. Cor. 3, $PC:PO::CD^2:CA^2$, that is, $\sqrt{PC}:\sqrt{PO}::CD:CA = 5(a+x) \div 8\sqrt{a}$ the semitransverse. Hence the area of the ellipsis being as $CD \times CA$, we have $((a^2 - x^2) \div 2a) \times (5(a+x) \div 8\sqrt{a})$, or $(a^2 - x^2)(a+x) = a \max.$ which, being fluxed and reduced, gives $x = \frac{1}{3}a$, and the area = $\cdot 314159 \times 10a \sqrt{a} \div 27 = (80 \div 27) \times \cdot 314159 = 9\cdot 308422677$.

Again, let EFMP be the inscribed semiparabola, and MP an ordinate to the conjugate axis DE, of the ellipsis. Put $a = CA$, $b = CD$, and $x = CP$. Then, by *Emerson's Conics*, 7, I. $CD^2:CA^2::DPE:PM^2$, that is, $(a \div b) \sqrt{(b^2 - x^2)} = PM$. Hence, the area of the parabola being as $EP \times PM$, we have $(b+x)(a \div b) \sqrt{(b^2 - x^2)}$, or $(b+x) \sqrt{(b^2 - x^2)} = a \max.$ In fluxions &c. $x = \frac{1}{3}b$, and the area = $ab \sqrt{3} = 5\sqrt{3} = 8\cdot 660254038$.

N. B. The area of the parabola will be the same if the vertex be at A instead of E.

XV. QUESTION 171, answered by Mr. Gregory, the proposer.

It has been shewn, by writers on optics, that when the apparent altitude of the sun's upper limb is equal to, or exceeds, $54^\circ 7'$ there can be no secondary rainbow, and that when it equals or exceeds $42^\circ 2'$ there can be no primary rainbow. We have, therefore, given, in the first case, $54^\circ 7'$ apparent altitude of the sun's upper limb, $-41''$ refraction $+ 5''$ parallax, $-15' 47''$ sun's semidiameter, or $53^\circ 50' 37''$, true altitude of the sun's centre, latitude $52^\circ N$, declination on July 1st, $23^\circ 8' 22'' N$, the complements of which form the three sides of a spherical triangle, to find one of the angles, namely the hour angle from noon, $28^\circ 27' 12''$ equivalent to 1h. 53m. 59s, which doubled gives 3h. 47m. 38s. for the time during which no secondary bow can be seen. In the second case, we have $42^\circ 2'$ for the apparent altitude of the sun's upper limb, which corrected for the semidiameter ($-15' 47''$), refraction ($-1' 3''$), parallax ($+ 6''$) gives $41^\circ 45' 16''$ true altitude of the sun's centre; whence, taking the latitude and declination as before, we get 51° for the hour angle, equivalent to 3h. 24m. and twice this or 6h. 48m. is the time during which there can be primary bow.

In the above solution, no notice has been taken of the change of declination during the respective intervals: and it has been supposed.

posed that the meridian of the place does not vary considerable from that of Greenwich.

The same, answered by Mr. Wm. Marrat, Boston.

By *pa. 383, Vol. II. Art. 50, Martin's Philosophica Britannica*, when the alt. of the sun is $= 42^{\circ} 02'$ the summit of the interior bow will be depressed below the horizon, also when the sun's alt. is $= 54^{\circ} 10'$ the summit of the exterior bow will be below the horizon. Therefore we have given the lat. $= 52^{\circ}$, decl. $= 23^{\circ} 08'$, and the sun's alt. $= 42^{\circ} 02'$ and $54^{\circ} 10'$ to find the hour \angle s or times from noon. Now, when the sun's alt. is $42^{\circ} 02'$ the time from noon is found $= 3h. 22m. 8s.$ which doubled gives $6h. 44m. 16s.$ $=$ the length of time during which there can be no interior bow, and when the sun's alt. is $54^{\circ} 10'$ the time from noon is $= 1h. 51m.$ the double of which is $3h. 42m.$ $=$ the length of time during which there can be no exterior bow.

The same, answered by Merones Minor.

By Emerson's Optics 26, IV. *pa. 237, &c.* we have the greater radius of the primary bow $= 42^{\circ} 17'$, and that of the secondary bow $= 54^{\circ} 22'$, and the depression of the centre of the bow $=$ the alt. of the sun. Therefore, having the decl. $23^{\circ} 9'$, alt. $42^{\circ} 17'$, and lat. 52° , I find, by spherics, the hour \angle $50^{\circ} 7'$ before noon; in like manner with the decl. $23^{\circ} 8'$ I find the hour \angle $50^{\circ} 6'$ afternoon; and their sum $= 100^{\circ} 13' = 6h. 40m. 52s.$ the time required for the primary bow. Again, taking the declinations $23^{\circ} 8\frac{2}{3}'$ and $23^{\circ} 8'$, the alt. $54^{\circ} 22'$ and lat. 52° I find the hour \angle s $27^{\circ} 18\frac{2}{3}'$ and $27^{\circ} 17\frac{1}{3}'$, and their sum $= 54^{\circ} 36' = 3h. 38m. 24s.$ the time required for the secondary bow.

The same, answered by Tyro Philo Mathematicus.

From the principles of optics we learn, that, if the sun's apparent altitude be greater than $42^{\circ} 2'$, there cannot be any primary rainbow, and if greater than $54^{\circ} 7'$ there cannot be any secondary rainbow. Conseq. there are given the lat $= 52^{\circ}$, sun's dec. $23^{\circ} 8'$, and alt. of the sun, in the first case, (allowing for refraction. $1' 3''$ and parallax $7''$) $= 42^{\circ} 1' 4''$, and in the second case, (allowing for refraction. $41''$ and paral. $5''$) $= 54^{\circ} 6' 24''$, to find the hours, which come out $3h. 22m. 10s.$ and $1h. 51m. 27s.$ from noon, these doubled give $6h. 44m. 20s.$ and $3h. 42m. 54s.$ the intervals for the primary and secondary bows respectively.

This question was also answered by Messrs. Burdon, Lowry, May, and Thornoby.

XVI. QUESTION 172, answered by Mr. W. Peacock,
Birmingham.

Fig. 440, Pl. 23. Suppose ACB the Δ required, HE a diam. of the circumscribing circle drawn \perp to the base, O the centre and OD the rad. of the given infer. circle. Put $a = EH$, $b = OD$, and $x = EF$, then $FH = a - x$, and by *Prop. II. Cor. 2*, and *Prop. III. Cor. 7, Art. II. Vol. II. of the Repository*, $AF^2 : OD \cdot (2EF + OD) :: FH \cdot EH : AC \cdot CB =$, by the quest. to $4AF^2 = 4FE \cdot FH$, hence $AF^2 : OD \cdot (2EF + OD) :: EH : 4FE$, that is, in species, $(a - x) x : b (2x + b) :: a : 4x$, therefore $4a^2 - 4x^2 = 2abx + ab^2$, and consequently $x^2 - ax + \frac{1}{4}abx + \frac{1}{4}ab^2 = 0$. From this equation EF becomes known, and from thence the sides and base of the Δ are readily determined.

Mr. Blackwell sent a true solution to this question.

XVII. QUESTION 173, answered by Mr. James Cunliffe.

ANALYSIS. Fig. 441, Pl. 23. Suppose the thing done, and the lines drawn, as directed by the question. BgC and BAC being right \angle s, a semicircle whose diam. is BC will pass through g and i, Euc. 31, III, and $\angle gkB + ghi = gkB + gCB =$ a right \angle , Euc. 22, III, theref. $\angle ghi = gCB = DCB$. For the same reason a semicircle whose diam. is DB will pass through i and g, and $\angle giB (gih) = gDB = DCB$, Euc. 31, III, therefore the $\Delta s igh$ and DBC are similar. And in the same way it may be shewn, that the $\Delta s ieh$ and DAC are similar, and consequently the trapeziums eighe and ACBDA; therefore $ie = ig$. By reason of the sim. $\Delta s igh$, DBC, $DB : BC :: ig : gh$, also by the sim. $\Delta s ieh$, DAC, $DA (DB) : AC :: ie (ig) : he$, theref. $DB^2 : AC \cdot BC :: ig^2 : gh \cdot he$, and theref. by equating the products of the extremes and means, $DB^2 \cdot gh \cdot he = AC \cdot BC \cdot ig^2$; multiply both sides of the equation or expression by $AC \cdot BC \cdot ig^2$ and it becomes $DB^2 \cdot gh \cdot he \cdot AC \cdot BC \cdot ig^2 = AC \cdot BC \cdot BD \cdot DA \cdot ie \cdot ig \cdot gh \cdot he = AC^2 \cdot BC^2 \cdot ig^4$ which is to be a max. by the question, therefore $AC \cdot BC \cdot ig^2$ must also be a max. From what is deduced in the preceding part of the Analysis, it will easily be perceived, that the $\Delta s iEg$ and igh are sim. therefore $iE : ig :: ig : ih$, therefore $iE \cdot ih = ig^2$. Now, produce the $\perp Di$, till it meets the circle again in H, and draw CG \perp to it.—DH is a diam. of the circle, and it is well known that $Di \cdot GH = iE \cdot ih = ig^2$. Also, by Theo. 25, B. III. Simp. Geo. $AC \cdot BC = DH \cdot iG$. Therefore $AC \cdot BC \cdot ig^2 = DH \cdot iG \cdot Di \cdot GH$ must be a max. from what is done above, that is, $iG \cdot Di \cdot GH$ must be a max. because the diam. DH is given. Therefore the problem is evidently reduced to this, viz.

To divide the given diam. DH into three such parts, that the solid contained under them, may be the greatest possible, which it is well known, will be the case, when Di, iG, and GH are equal to each other, that is, when each is equal one third of DH. Whence the trap. ABCD may be easily constructed.

Remark. Because Ai and GC are equal to each other, it is evident that AC must pass through the centre of the circle and be a diam. thereof. And as Ae is \perp to DC the points D and e will coincide. Also, because eh is generally parallel to DB, the points B and h must also coincide.

The same, answered by Mr. Lowry.

ANALYSIS. Fig. 442, Pl. 23. Suppose the problem really solved, and let the several lines be drawn as directed by the problem. The diam. DE, being drawn through D, will evidently pass through i the middle of the diagonal, join CE and draw CF \perp to DE, and on DC demit the \perp s hm, ik. Then, it is well known, that the trapeziums ABCDA, eigh are sim. and that EC is = to the diam. of a circle circumscrib. the trap. eigh. And, by Euc. C. VI.

$$AC \cdot CB = DE \cdot Fi, AD \cdot DB = DE \cdot Di, eh \cdot hg = EC \cdot hm, \text{ and } ei \cdot ig = EC \cdot ik.$$

But, by sim. Δ s. $EC : EF :: Ch = Fi : hm$.

$$\text{Hence } EF \cdot Fi = EC \cdot hm = eh \cdot hg.$$

In the same way it is shewn, that $EF \cdot Di = ei \cdot ig$. Hence $AC \cdot CB \cdot BD \cdot DA \cdot ie \cdot ig \cdot gh \cdot he$ is = to $DE^3 \cdot Fi \cdot Di \cdot EF^2$, which, by the question, is to be a max: but the diam. DE is a constant quantity, therof. $Fi \cdot Di \cdot EF^2$, or the solid $Fi \cdot Di \cdot EF$ must be a max. and this, it is well known, will obtain, when, the parts Fi, Di, and EF are equal. The method of constr. is therof. evident.

COR. Since $Di = EF$, the chords iB, FC are equal, therof. AC will be a diam. of the circle, and the point h will coincide with B, and the point e with D.

The same, answered by Mr. J. H. Swale.

ANALYSIS. Fig. 442, Pl. 23. The problem being supposed solved, and the lines drawn as directed by the question, produce Di to meet the circle at E, join EC and demit the \perp s CF, ik, hm, upon DE and DC respectively. Then since $AD = DB$, and Di \perp to AB, we have Ai = iB, therefore DE passes through the centre O, and is equal to the given diameter.

Now $AC \cdot CB \cdot AD \cdot DB \cdot ie \cdot ig \cdot gh \cdot he$, is to be a max. or, which is the same, $AD^3 \cdot ie \cdot AC \cdot CB \cdot gh \cdot he$, is to be a max. for $AD = DB$.

DB, and ie is = to ig , by *Harrison's Propositions in the Repository*.

Again $AC \cdot CB = DE \cdot Ch$, $AD^2 = DE \cdot Di$, $ie^2 = EC \cdot ik$, and $gh \cdot he = EC \cdot hm$; for by *Prop. 15. Pa. 235, Vol. II. Repository*, EC is = the diam. of the circle circumscrib. the trap. $ighe$, therf. when the original folid is the greateft, the produft $DE^2 \cdot EC \cdot ik \cdot hm \cdot Ch \cdot Di$ will be the fame, that is $EC^4 \cdot Di^2 \cdot Ei^2$ muft be a *max.* fince, by fim. Δs $DE \cdot ik = Di \cdot EC$, and $DE \cdot hm = Ch \cdot EC$.

Moreover, becaufe $EC^4 \cdot Di^2 \cdot Ei^2$ is a *max.* the produft $EC^2 \cdot Di \cdot Ei$ is likewife a *max.* or, becaufe $ED \cdot EF = EC^2$, and ED conftant, the folid $EF \cdot Fi \cdot iD$ muft be a *max.* which will evidently obtain, when $EF = Fi = iD$, each being = to one third of the diam. DE . Hence the conft. is evident.

COR. The angles ABC , ADC are right angles.

Mr. Fletcher, the propofer, alfo answered this queftion.

XVIII. QUESTION 174, answered by Mr. John Lowry, Birmingham.

Put r = the radius, and x = the tangent of the mean arc; then, by the propofition, $(a \div b)x^2$ will represent the tangent of the greater arc. And by *Prop. IX. of Emerson's Trigonometry*, $r^2 + (a \div b)x^2 \div r^2 :: (a \div b)x^2 - x : r^2(ax^2 - bx) \div (rb^2 + ax^3)$, the tangent of the difference of the fecond and third arcs, therefore, when the difference of thofe arcs is the greateft poffible, $(ax^2 - bx) \div (rb^2 + ax^3)$ muft be a *max.* This expreffion put into fluxions and reduced gives $b^2r^2 + a^2x^4 = 2abr^2x + 2ab^2x^3$, or $r^2 + (a \div b)x^2 = (2a \div b)x(r^2 + x^2)$. Multiply each fide of this equation by r^2x and divide by $r^2 + (a \div b)x^2$ and $r^2 + x^2$, and it becomes $r^2x \div (r^2 + x^2) = 2r^2(a \div b)x^2 \div (r^2 + (a \div b)x^2)$. Now, by *Prop. II. Schol. I. ibid.* $2r^2x \div (r^2 + x^2)$ and $2r^2(a \div b)x^2 \div (r^2 + (a \div b)x^2)$ are the fines of twice the arcs, whole tangents are x and $(a \div b)x^2$ refpectively. Therefore the fine of twice the greater arc is equal to half the fine of twice the mean arc, as it fhould have been printed in the queftion.

The fame, answered by Mr. John Surtees.

Let $\text{rad.} = 1$, $b \div a = n$, and the tangents of the other two arcs nx , and nx^2 . Then the tangent of the difference of thofe arcs will be $(nx^2 - nx) \div (1 + n^2x^3) = a \text{ max.}$ by the queftion, which put into fluxions and reduced gives $2nx^2 \div (1 + n^2x^3) = nx \div (1 + n^2x^3)$.

(1 + $\pi^2 x^2$.) Hence it appears, that the difference of those arches must be a maximum when the sine of twice the greater arc is equal to half the sine of twice (not *thrice* as per quest.) the mean or the less of the two unknown arches.

The same, answered by Mr. Geo. Sanderfon.

Let A, B, and C represent the three arches,

$d = \frac{a}{b}$ the tang. of A, one extreme

T = tang. of B the mean, and

t = tang. of C the other extreme, radius 1.

Then $\frac{T-t}{1+tT} = \text{tang. of } B-C$, (Simpson's Trigon. pa. 55.)

which will be a maximum, when B - C is a maximum.

$$\text{Or, because } \frac{T^2}{d} = t, \quad \frac{T - \frac{T^2}{d}}{1 + \frac{T^2}{d}} = \frac{dT - T^2}{d + T^2} = a \text{ max.}$$

whose fluxion made = 0, and the equation reduced, we have

$$dT - 2dT - 2dT^2 + T^4 = 0; \text{ and by subst. } \frac{T^2}{t} \text{ for its equal}$$

$$d, \frac{T^4}{t^2} - \frac{2T^2}{t} - \frac{2T^4}{t} + T^4 = 0; \text{ whence, by proper reduction,}$$

$(T^2 + 1) \times 2t = (t^2 + 1) \times T$. That is, the square of the secant of B, multiplied by twice the sine of C and divided by the cosine of C, is equal to the square of the secant of C, multiplied by the sine of B divided by the cosine of B. But the square of

the secant is = $\frac{\text{radius}}{\text{square cosine}}$; whence, by putting m and n for the

sine and cosine of B, and s and c for those of C, we have $\frac{1}{n^2} \times$

$\frac{2s}{c} = \frac{1}{c^2} \times \frac{m}{n}$; which reduced, $2sc = mn$; but by Simpson's

Trig. page 56, the sine of the double of any arch is equal to twice the rectangle of the sine and cosine of the single arch. Therefore (2sc) the sine of twice C, the extreme arch, is equal to (mn) half the sine of twice B, the mean, as it ought to have been printed in the question.

XIX. QUESTION 175, answered by Mr. Lowry, Birmingham.

CASE I. When the sum of the squares of the sides, together with the square of the perpendicular, is given. Fig. 443, Pl. 23.

CONSTRUCTION. Bisect the given base AB at D, and draw BC to make the $\angle ABC =$ to the given one: draw DI \perp to BC, and DH \perp to AB, also divide BD at E so that EB : ED :: $2BH$: IH, and let the rect. $2BD \cdot DG$ be taken $=$ to the diff. between the given space and twice the square of AD or BD, and from E, to BC, apply EC $=$ to a mean proportional between EG and EB; join AC, so shall ACB be the Δ required.

DEMONSTRATION. Produce DC to meet a circle described through the three points B, C, G, at Q, and join QG, QB; draw DK \parallel to EC to meet BC produced at K, and on AB demit the \perp CP. Then since $EC^2 = BE \cdot EG$, by constr. EC is a tangent to the circle at C, therf. Euc. 32. III. the \angle s ECB, BGC are $=$ but DK is \parallel to EC, therefore the \angle s DKB, ECB, are $=$, and so the \angle s DKB, BGC or its $=$ BQD, are $=$, and, therefore the points D, K, Q, B are in the circumference of the same circle; hence

$BC \cdot CQ = BC \cdot CK$, wherefore

$DC \cdot CQ (BC \cdot CK) : BC^2 :: CK : BC$: but by sim. Δ s and constr.

$CK : BC :: ED : EB :: IH : 2BH$, therefore

$DC \cdot CQ : BC^2 :: IH : 2BH$

$:: IH \cdot BH (=DH^2) : 2BH^2 :: PC^2 : 2BC^2$; hence

$2DC \cdot CQ = PC^2$. To each of these equals add twice the square of DC, then

$2DC \cdot DQ$ or, its $=$, $2BD \cdot DG$ is $= PC^2 + 2DC^2$; but

$AC^2 + BC^2 = 2AD^2 + 2DC^2$, therefore

$AC^2 + BC^2 + PC^2 = 2AD^2 + 2BD \cdot DG$: but by constr.

$2BD \cdot DG =$ to half the diff. between the given space and $2AD^2$;

therefore $AC^2 + BC^2 + PC^2$ is $=$ to the given space. Q. E. D.

CASE II. When the difference of the squares of the sides together with the square of the perpendicular is given.

Fig. 444. Pl. 23.

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CONSTRUCTION. Bisect the given base AB at D and draw BC to make the $\angle ABC =$ to the given one; draw $DI \perp$ to BC, and $DH \perp$ to AB, and let the rect. $HI \cdot IQ$ be taken $=$ to the given space, and take $DV : DB :: IB : IH$, and $SD = {}_4VD - {}_2DB$. By Euc. 29. VI. cut BD at P so that the rect. $SP \cdot PD$ may be $=$ to the excess of the rect. $BI \cdot IQ$ above the square of DB, and draw $PC \perp$ to AB and let it meet BC at C; join AC, so shall ACB be the Δ required.

DEMONSTRATION. Since by construction, $SD = {}_4VD - {}_2DB$, the rect. $SP \cdot PD$ is $= ({}_4VD - {}_2DB) \cdot PD + PD^2$, to each of these equals add the square of DB, then Euc. 4. II.

$SP \cdot PD + DB^2 = {}_4VD \cdot PD + PB^2$: but by construction,

$SP \cdot PD + DB^2 = BI \cdot IQ$, therefore

${}_4VD \cdot PD + PB^2 = BI \cdot IQ$. Again by similar triangles,

$PC^2 : PB^2 :: DH^2 (BH \cdot HI) : DB^2 (BH \cdot BI)$

$:: HI : BI :: DB : DV$, by conf.; hence

$PC^2 : {}_4DB \cdot PD :: PB^2 : {}_4DV \cdot PD$, therefore by composition,

$PC^2 + {}_4PDB : PC^2 :: PB^2 + {}_4PDV (= BI \cdot IQ) : PB^2$; conseq.

$PC^2 + {}_4DB \cdot PD : BI \cdot IQ :: PC^2 : PB^2 :: HI : BI$

$:: HI : IQ :: BI : IQ$, therefore

$PC^2 + {}_4PDB = HI \cdot IQ$: but ${}_4PDB = AC^2 - BC^2$, therefore

$AC^2 - BC^2 + PC^2 = HI \cdot IQ$, i.e. $=$ the given space, by constr.

Q. E. D.

The same, answered by Mr. J. H. Swale.

ANALYSIS. Fig. 443, Pl. 23. Suppose ACB the Δ required, let D be the middle of the base, and join DC, and drop the \perp CP. Then, in the first case, when $AC^2 + CB^2 + CP^2 (= {}_2DC^2 + CP^2 + {}_2DB^2 = {}_3CP^2 + {}_2DP^2 + {}_2DB^2)$ and the $\angle BAC$ are given, ${}_3CP^2 + {}_2DP^2$, and the ratio of $AP^2 : CP^2$ will be given, let the latter be equal to $m : n$, then $3n \times AP^2 + 2m \times DP^2$, or $(2m + 3n) \times DP^2 + 6n \times AD \cdot DP$ will be given, and the construction is evident from 29. VI. of *Playfair's Edition of Euclid's Elements*. And, in the second case, when $AC^2 - CB^2 + CP^2 (= AP^2 - BP^2 + CP^2 = CP^2 + {}_2AB \cdot DP)$ and $CP^2 : AP^2 (= n : m)$ are given, we shall have $n \times AP^2 + 2m \times AB \cdot DP$, or $n \times DP^2 + (4m + n) \times AD \cdot DP =$ to a given space. Hence the construction is easily effected by means of the Prop. above cited.

Mr. Louis Hill also constructed this Problem.

XX. QUESTION 176, answered by Mr. W. Burdon, *Acafter-Malbis, York.*

ANALYSIS. Fig. 445, Pl. 23. Suppose ACB to be the Δ required, and CF the diameter of its circumscribing circle; join AF and let it be produced to meet the \perp CP, produced, in D; draw FK \perp to AB and join FB, FE. Now, the $\angle AFC$ being $= \angle ABC$, and the $\angle FAC = \angle FBC$, $=$ a right angle, the $\Delta s AFC, FBC$, are sim. Again, the $\angle s ACD, FCB$, standing upon $=$ arches, are equal, and the $\angle DAC = \angle FBC =$ a right angle, therefore the $\angle FDC = \angle CFE$, and the $\Delta s FCE, DCF$ are similar. Therefore $AC : CB :: CF : CE :: CD : CF$, but CF and the ratio of $AC : CB$ are given, therefore CD and CE are given. Hence $BP(AK) : BK (AP) :: PG : KF (PE) :: KF (PE) : PD$, therefore $PG \cdot PD = PE^2 =$ a given space; but DG is given; therefore PD will be found by Prop. 29, B. VI. *Professor Playfair's Edit. of the Elements.*

The same, answered by Mr. John Lowry.

CONSTRUCTION. Fig. 445, Pl. 23. Take CD to be the given diameter, as the greater side to the less, and in CD take CG a 3rd proportional to CD and the given diameter; make the rect. GPD $=$ to the square of the given prolongation of the \perp , and through P draw APB \perp to CD to meet semicircles described on CD and CG, as diameters, at A and B; join AC, BC, and ACB is the Δ required.

DEMONSTRATION. About the ΔABC let a circle be described, and draw BG to meet it at F, and join CF, DA, AF, and EF, E being the point where the circle cuts CD. Then because the $\angle FBC$, in the semicircle GBC, is a right angle, CF is the diameter of the circle ACB. therefore the $\angle CAF$ is a right angle, and the $\angle CAD$ is in the semicircle, by constr. therefore the straight line joining the points A and D passes through the point F. Again, FE is evidently parallel to AB, therefore the arcs AF, EB are equal, and therefore the $\angle s ACF, BCE$ are also equal; therefore the right angled $\Delta s ACF, BCG$ are equiangular, and so are also the $\Delta s FCD, FCG$: hence $CG : CF :: CF : CD$, but by constr. $CG : \text{diam.} :: \text{diam.} : CD$, consequently CF is $=$ the given diam. Moreover $AC : CB :: CF : CG :: CD : CF$, that is in the given ratio by constr. Again, by the circle, $AP : PC :: PE : PB$, or $AP^2 = CP \cdot PD : CP^2 :: PE^2 : PB^2 = CP \cdot PG$, therefore $PE^2 = PD \cdot DG =$ the square of the given prolongation by constr.; therefore PE is $=$ to the given prolongation of the perpendicular.

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This question was likewise answered by Messrs. Cunliffe, Hill, Johnson, Squire, and Swale.

XXI. QUESTION 177, answered by Mr. Lowry, the proposer.

Most of our ingenious correspondents, who attempted to answer this question, founded their solutions on a principle, which, to us, seems not true, viz. that, "*The base of the greatest triangle, which can be inscribed in a given circle, divides the diameter, drawn from its vertex, into segments which have the given ratio of 3 to 1.*" This is undoubtedly true, in some particular cases, but not generally so, nor is it so, we presume, in the instance before us. The problem does not appear to us to admit of a GEOMETRICAL CONSTRUCTION, and our opinion is corroborated, by the following *fluxionary* solution, which we have received from the proposer.—Suppose ACB (fig. 446, pl. 23,) the required Δ , BD the \perp , and O the centre of the circumscribing circle; draw the diam. EG \perp to the base, and draw BS \perp to EG, draw also the radii OB, OC. Put $4a$ = the given diff. of the squares of the sides, r = OC the rad. of the given circle, and x = half the base = FC. Then by a well known theorem $AB^2 - BC^2 = AC \cdot 2FC$ hence $FD = BS = a \div x$, and by Euc. 47. I, $\sqrt{r^2 - x^2} = OF$ and $\sqrt{r^2 - a^2} \div x^2 = OS$, therefore $BD = FS = \sqrt{r^2 - x^2} + \sqrt{(x^2 r^2 - a^2)} \div x^2$, and the area of the Δ ABC is $= x \sqrt{r^2 - x^2} + \sqrt{x^2 r^2 - a^2}$, which, by the question must be a *maximum*. This expression, put into fluxions, gives $(r^2 x - 2xx) \cdot (r^2 - x^2)^{-\frac{1}{2}} + r^2 x \cdot (x^2 r^2 - a^2)^{-\frac{1}{2}} = 0^2$, and, by reduction, we have $r^2 x^3 (r^2 - x^2) = (2x^3 - r^2) \cdot (x^2 r^2 - a^2)$, which being further reduced produces $x^6 - (3r^2 + 4a^2) \div 4 \cdot x^4 + (r^2 \div 4 + a^2) \times x^2 = r^2 a^2 \div 4$, whence the base AC becomes known.

COROLLARY. It is evident, from what is done above*, that $BD : OS :: 2FC : OC$, when the triangle is a maximum.

XXII. QUESTION 178, answered by Mr. John Andrew, Carh.

ANALYSIS. Fig. 447, Pl. 24. Let ACB be the Δ required, AB the given base and D, P, the two given points therein; let ACB be the circumscribing circle and O its centre. Let fall the \perp OG. draw the diam. COF, and join OB. Then $AG = GB$ (Euc. 3. III.), and it is evident that DG, GP are given by the question, therefore, by sim. Δ s $DG : GP :: DO : OC = OB$: Hence this

CONSTRUCTION. With the two given segments DG , GB , and DO , OB in the same ratio of DG to GP , make the $\triangle DOB$ by *Prop. XXII. Appendix to Simpson's Algebra*, and produce DO till $OC = OB$, also produce BD till $AG = GB$; join AC , BC , and ACB will be the \triangle required.

The same, answered by Mr. Burdon.

ANALYSIS. Fig. 447, Pl. 24. Suppose the thing done, ACB the required \triangle , CF the diam. of the circumscribing circle, and D , P , the given points. Bisect the base AB with the $\perp GO$, and draw $Of \parallel$ to AB . Then by *sim. \triangle s*, $OC:DC::Of:DP$, or $FC = 2OC:DC::2Of:DP$, *theref. dividendo*, $DF (= FC - DC):DC::PG - DG (2Of - DP):DP$; but the points D , G , P , are given, therefore the ratio $DF:FC$ is given. Hence this

CONSTRUCTION. Having bisected the given base AB in G , take $DL:DP::GP - DG:DP$, apply DM such that $LD \cdot DM = AD \cdot DB$, and upon DM describe a semicircle cutting the $\perp PC$ in C ; join AC , CB and ABC will be the \triangle required.

DEMON. By *sim. \triangle s* $DL:DP::DF:DC$, in the given ratio; and $DL:DF::DC:DM$, therefore $DF \cdot DC = DL \times DM = AD \cdot DB$, by construction. Q. E. D.

The same, answered by Mr. James Cunliffe.

ANALYSIS. Fig. 447. Pl. 24.) Let ACB , the required \triangle , be circumscribed by a circle, the diam. Nn being \perp to the base AB , which it therefore bisects in G (*Euc. 3. III.*); make the chord $CI \parallel$ to Nn , intersecting AB in P , and draw the diam. CF to cut AB in D ; also make $FL \parallel$ to Nn , and join FI ; draw $CM \perp$ to CF , meeting AB produced in M . The $\angle CIF$ is a right angle (*Euc. 31. III.*), whence FI is \parallel to AB and is bisected by Nn in g (*Euc. III. 3.*) therefore $gI = GP = GL$.—Now the \triangle s DLF , DCR , are manifestly similar, therefore $DL:DF::DC:DM$, whence $DL \times DM = DF \times DC$ (*Euc. 16 VI.*), also $AD \times DB = DF \times DC$ (*Euc. 35. III.*); therefore $DL \times DM = AD \times DB$: but AD , BD and DL are all given by the question, therefore DM becomes known, and the following Construction is suggested.

Bisect the base AB in G , and therein take $GL = GP$, on DB take DM a 4th proportional to DL , DA , and DB ; then upon DM as a diam. describe a semicircle, cutting a \perp to AB , drawn through P , in C ; join AC , BC , and ACB is the \triangle required.

The

The same, answered by Mr. Louis Hill.

CONSTRUCTION. Fig. 448, Pl. 24. Let AB be the given base, and D, P , the given points therein. On DP describe a semicircle, and from A and B draw tangents thereto, as AQ, BO . On AB form the $\triangle ACB$, so as to have its vertex in the line PC , drawn \perp to AB , and its sides AC, CB in the given ratio of AQ to BO , then is the $\triangle ACB$ the required one.

DEMONSTRATION. Through the points D, P, C , describe a circle to intersect AC at G and BC at F , join GF , and produce CP, CD , to meet the circumscribing circle at I and E ; then by constr.

$AC : CB :: AQ : BO$, or

$AC^2 : CB^2 :: AQ^2 (= DAP) : BO^2 (= PBD) : \text{but}$

$DAP = GAC$, and $PBD = FBC$; hence

$AC^2 : CB^2 :: GAC : FBC$; therof.

$AC : CB :: AG : FB$, therof GF is \parallel to AB , and therefore the arcs GD, PF are equal, and, so are also the arcs AE, BI ; therefore EI is \parallel to AB , and the $\angle EIC$ a right \angle ; consequently CE is the diam. of the circle circumscribed about the $\triangle ABC$, and passes through the given point G , as required.

The same, answered by Mr. J. Johnson.

ANALYSIS. Fig. 447, Pl. 24. Suppose it done, and that ACB is the \triangle required, D and P the given points in the base AB , which is also given. Make $FL \perp$ to AB , then $AL = BP$, therefore the point L is given: but LD and DP are given lines, hence by sim. $\triangle s$ $LD : DF :: DP : DC$, therefore the ratio of $FD : DC$ is given: but the rect $FD \cdot DC = AD \cdot DB$ is a given rectangle; therefore there are given the ratio and the rect. of two lines FD, DC to determine them. The construction is easy from the Analysis.

The same, answered by Mr. Lowry.

CONSTRUCTION. Fig. 447, Pl. 24. Let AB be the given base, and D and P the given points therein. On AB describe a semicircle, and let the $\perp s$ PK, DH , let fall from P and D , meet it at K and H ; draw $KE \parallel$ to AB , and from D , to KE , apply DE a third proportional to PK and DH , produce ED to meet KP continued in C ; the vertex of the required $\triangle ACB$.

DEMONSTRATION. Through the points A, C, B let a circle be described, intersecting CK at I and CE at F , then we have
only

only to prove that EDC is the diam. of the circle ACBF; for the \perp CP passes through the given point by constr. Join FI, then by sim. Δ s,

DC : CP :: DE : PK :: DE · PK : PK², but by construction

DE · PK = DH² (= AD · DB = DC · DF), and

PK² = AP · PB = CP · PI; hence

DC : CP :: DC · DF : CP · PI :: DF : PI,

consequently FI is \parallel to AB, and the \angle FIC is a right angle; hence FC is the diam of the circumscribing circle and passes through the given point G.

The same again, by Mr. J. H. Swale.

Fig. 447, Pl. 24. Imagine ACB the Δ to be found, and that the diam. of its circumscribing circle, drawn from the vertex, meets the base at D. Then it is very evident that the ratio of DO to OC, = the ratio of DG to GP, is given; consequ. the ratio OC — OD (= DF) to OC + OD (= DC) is given: but the rect. FD · DC = AD · DB is likewise given. Hence, we have given the ratio and the rect. of the two lines FD, DC to determine them, the method of doing which is well known. These lines then being determined, draw PC \perp to AB and to it apply DC of the given length; join AC, CB, and it is done.

Mr. Nicholson sent a neat solution to this question.

XXIII. QUESTION 179, answered by Mr. Lowry.

CONSTRUCTION. Fig. 449, Pl. 24. On the indefinite straight line IC, take AE such, that its square may be equal to the rectangle of the given differences, and let the solid AQ · AE² be taken equal to the given difference of the cubes of the sides. Draw KE \perp to AE and equal to a mean proportional between AE and a third part of EQ; bisect AE at L, and, with the centre L, and distance LK, let a circle be described to meet IC at I and C. With the lines AC, AI and the given difference (CH) of the segments, made by the \perp , form the Δ ACH, and draw AB \parallel to HC, and to it apply CB = AH or AI: so shall ACB be the Δ required.

DEMONSTRATION. Through the points A, B, C describe the circumference of a circle, draw the diam. FG \perp to the base AB and join FC. Then AE is = to the diff. of the sides AC, CB, and $2CP = HC =$ the given diff. of the segments made by the \perp . But, by a well known property, the square of the diff. of the sides is = to four times the rect. GP · GD, that is, AE² = 4GP · GD = the rect of the given differences, by constr.; therefore $2GD$,
that

that is, the diff. of the segments of the base, made by (CD) the line bisecting the vertical angle, is equal to the given diff. Again, by Prop. E. Art. XXII. Vol. II. of the Repository, the difference of the cubes on the sides AC, CB, or on AC, EC is equal to the cube on AE together with thrice the solid $AE \cdot AC \cdot EC$; but $AC = IE$, therefore $AC \cdot EC = IE \cdot EC = KE^2$; therefore thrice the solid $AE \cdot AC \cdot EC$ is $= AE \times 3KE^2 =$ (by constr.) $AE \cdot EQ$. Conseq. the diff. of the cubes on AC, CB is $=$ the cube on AE together with the solid $AE \cdot EQ$, that is, $=$ to the solid $AE \cdot AQ =$ the given diff. of the cubes. Q. E. D.

The same, answered by Mr. James Cunliffe.

ANALYSIS. Fig. 449, Pl. 24. Suppose ACB the required Δ , G the middle point of the base AB. Draw $CD \perp$ to AB, and bisect the $\angle ACB$ by the line CP meeting AB in P. GD, GP are each given by the question, and it is well known that $\frac{1}{2}(AC^2 - BC^2) = GD \cdot GP$, from whence $AC - BC$ becomes known. But $AC^2 - BC^2 = (AC - BC) \cdot (AC^2 + AC \cdot BC + BC^2)$ is also given by the question, and $AC - BC$ has been found above, therefore $AC^2 + AC \cdot BC + BC^2$ becomes known. Again take $(AC - BC)^2$ from the preceding, and there will remain $3AC \cdot BC$, which also becomes known, and therefore $AC \cdot BC$ becomes known. Now the diff. and rect. of the sides being found, the sides themselves may easily be found, and the problem thereby readily constructed.

The same, answered by Mr. Richard Nicholson, Private Teacher of the Mathematics, at Liverpool.

ANALYSIS. Fig. 449, Pl. 24. Suppose the thing done ABC the required Δ , and G the middle of the base, and join GC. Then GP $=$ half the given diff. of the segments of the base, made by the \perp CP, and GD $=$ half the given diff. of the segments of the base, made by the line CG, bisecting the vertical angle. Now,

$AC^2 - BC^2 = (AC^2 + AC \cdot BC + BC^2) \times (AC - BC)$, but $AC - BC$ is given, its square being $= 4GDP$, by P. IV. p. 205, V. I; theref. $AC^2 + AC \cdot BC + BC^2$ is given: but, Sim. Geo. II. 11.

And Eu. I. 47,

$AC^2 + BC^2 = 2AF^2 + 2DC^2 + 2FD^2$; and by Sim. Geo. III. 26, and Eu. I. 5.

$AC \cdot BC = AF^2 + DC^2 + ED^2 - EF^2$, therefore

$AC^2 + AC \cdot BC + BC^2 = 3AF^2 + 3DC^2 - EF^2 + 2FD^2 + ED^2$; hence

$AF^2 + CD^2$ is given, conseq. $AC \cdot BC$ is given, therefore

$AC^2 + 2AC \cdot BC + BC^2$ is given, conseq. $AC + BC$ is given, theref. the

the sides are given : but $AC^2 - BC^2 = AB \times 2GP$ is given ; therefore AB is given and the construction is obvious.

Solutions were also received from Messrs. Andrews, Burdon, Hill, and Swale.

XXIV. QUESTION 180, answered by Mr. James Cunliffe.

GEOMETRICAL ANALYSIS. Fig. 450, Pl. 24. Let CSD and EOF , be the two given circles, O and S their centres, and P the given point in the line OS , and suppose PF the line required to be drawn, forming the chords CD and EF . Describe a circle concentric to CSD and equal to EOF , in OP produced take $Sp = OP$, and through p draw $pF' \parallel$ to PF cutting the last described circle in e and F' , and let CR and Sm' be drawn \perp to pF' , the latter cutting PD in n . It is plain that $eF' = EF$, and, by Euc. 3. III, the chords eF' and CD are bisected by the $\perp Sm'$ in the points m' and n . Draw the radius SC , which produce till it cuts pF' in T ; then by reason of the parallels $PD, pF', SP : Pp :: SC : CT$, and because SP, Pp , and SC are all given, CT from thence becomes known.—Moreover $CD + EF = CD + eF'$ is a given length, by the question, therefore their halves $Cn + m'F' = Rn' + m'F' = RF'$ must be half the said given length. Now TRC being a right angle, the locus of the point R will be the periphery of a semicircle whose diameter is the given line CT , as appears from Euc. 31. III, from which conclusions the following CONSTRUCTION is derived.

In the circle CSD draw any radius SC , which produce till CT is a 4th proportional to $SP, Pp = OS$, and SC ; upon CT , as a diameter, describe a semicircle:—then, by *Prop. 7th of Barrow's restitution of Apollonius on Inclinations*, draw the line TRF' , cutting the semicircle in R , and meeting the circle eSF' in F' , so that RF' may be equal to half the given sum of the chords, and draw $Sm' \perp$ to TF' . Again, upon PO as a diameter describe another semicircle, in which apply the chord $Om = Sm'$, and through m draw the line PmF , cutting the given circles in the points C, D , and E, F , and the thing is done, as appears from the Analysis.

If the diff. of the chords is given, the construction will be the same, only the line TR must be so drawn, by the prob. before referred to, that eR may be equal to half the said given difference.

The same, answered by Mr. Lowry.

GEOMETRICAL ANALYSIS, Fig. 451, Pl. 24. Let COB , FQE be the given circles and P the given point, in the line joining their centres O and Q , and suppose the line CPE to be really drawn

rawn, as required, viz. so that the sum or diff. of the chords CB, EF may be equal to a given line. On CE demit the \perp s OD, Qm, and describe the circle ILQ, concentric to EQF and equal to COB; join OB, OC, and QE, and draw QK \parallel to OB to meet PE at K, and the circle at I; draw IL \parallel to PE meeting Qm at H, and on PE demit the \perp s IN, Ln. Then, it is evident, (since QI is equal and parallel to OB) that the chords IL and CB are equal, and that FE and IL are bisected, by the \perp Qm, at m and H, therefore when the sum or difference of the chords FE, CB is given, the sum or diff. of Em, IH (Nm) is also given, that is EN or En is given. Now, by reason of the parallels, OP : OB :: PQ : QK, therefore QK is a given line, and QE is also given, therefore the diff. of their squares, or the diff. of the squares of Em, mK is given. Let a semicircle be described upon mE, and therein apply mG = to mK, and let ESs be drawn \parallel to Gm (or which is the same \perp to GE) to meet the line joining the points GN, Gn, at S and s. Then, by Euc. 47. I, the square of GE is = to the diff. of the squares of Em, mG (mK), that is, equal to a given space, therefore GE is a given line. Again by sim. Δ s IQ (OB) : QK :: mN (mn) : Km (Gm) :: NE : ES :: nE : Es, therefore ES or Es becomes known, according as the sum or diff. of the chords is given: Hence we derive the following

CONSTRUCTION. Take the line GE such that its square may be = to the diff. of the squares of QE and QK, (QK being taken a 4th proportional to OP, PQ, and OB). Draw SEs and Gm both \perp to GE, and take ES or Es a 4th proportional to OP, PQ, and half the given sum or diff. respectively, and draw GS, or Gs, and apply thereto EN, or En, equal to the said sum or diff. respectively, as the case may be, then, the intersections of these lines, with the \perp Gm, will determine the chords Em, mN (or mn), and this being done, the method of drawing CPE is evident.

If instead of the sum or difference, the rectangle of the chords had been given, the problem might have been constructed with nearly the same facility; for, by reasoning as above, it would appear, that the rectangle and diff. of the squares of Em, mK would be given, and therefore the chords might be determined by the 87th Prop. of Euclid's Data.

Or if the sum or diff. of the squares had been given, then the ratio, and the sum or diff. of the squares of mK, mN, would appear to be given, and the chords might be easily determined by the data.

But when the sum or difference of the squares of the chords is given, the problem may be done, in a more general manner, for the point, through which the required line is to pass, may be given

out

out of the line joining the centres, as will be evident from the following

ANALYSIS, Fig. 452. Pl. 24. Let OB , QF , OP , QP , and OQ , be drawn, and on OQ as a diameter, describe a circle to intersect OG , drawn \parallel to CE , at G , and join GQ ; on OG demit the \perp PK , and on CE the \perp OD ; produce KP till $KL = KG$, and draw GL to meet the circle at I ; join OL , IP , and PG . Then, since the sum or diff. of the squares of FE , CB , is given, the sum or diff. of the squares of their halves, FH , BD , is also given. Let us, in the first place, suppose that the sum of their squares is given; then, by a well known property of plane Δ s, the sum of the squares of OP , BD , is equal to the sum of the squares of OB , PB , and the sum of the squares of PQ , FH , is equal to the sum of the squares of FQ , PH ; therefore it is evident, that when the sum of the squares of FH , BD , is given, the sum of the squares of PH , PD , is given: but $PH = KG = KL$, and $PD = KO$, hence, the sum of the squares of PH , PD , is $=$ to the square of OL , hence OL is a given line. Moreover, since $KG = KI$, and the angle at K a right angle, the arc OI will be a quadrant of the circle $OIQG$, and I will be a given point; and since the angles KGL , GLK are equal, each being half a right angle, the angle PLI will be three fourths of two right angles, and consequently the locus of the point L will be a circle described through the points P , I , to contain the given angle PLI , and since O is a given point, and OL a given line, the point L will be determined, and the Construction manifest from the Analysis.

Again, when the diff. of the squares is given, it will appear, as before, that the diff. of the squares of OD (HG) and HQ , will be given: but the diff. of the squares of HG , HQ is $=$ to the diff. of the squares of PG , PQ , and PQ is a given line, therefore PG is given, and the method of Construction evident.

Messrs. Hill, Nicholson, and Swale, sent solutions to this question.

XXV. QUESTION 181, answered by Mr. Lowry.

CONSTRUCTION. Fig. 453, Pl. 24. Join PB , and draw PF to make the $\angle BPF =$ to the given \angle , and meet AC at F ; make the rectangle $PB \cdot PI =$ to the given one, and draw FG to make the $\angle PFG =$ to the $\angle PBD$, and, parallel thereto, draw PQ meeting AC in Q . Then, by Prob. V. B. I. of the determinate section, by Wales, cut CQ at E so that $CE \cdot EF : PF \cdot EQ :: PI : PQ$; join EP , and draw PD to make the $\angle DPE =$ to the given $\angle BPF$, and the thing is done.

DEMONSTRATION. By constr. the $\angle DPE =$ to the $\angle BPF$, that is, equal to the given angle. Also by constr.

$CE \cdot EF$

$CE \cdot EF : PF \cdot EQ :: PI : PQ$, and by sim. Δ s

$EF : EQ :: FG : PQ$, therefore

$CE \cdot EF : PF \cdot EQ :: FG \cdot CE : PQ \cdot PF :: PI : PQ$

$:: PI \cdot PF : PQ \cdot PF$, hence,

the rectangles $FG \cdot CE$, $PI \cdot PF$ are equal, consequently

$CE : PI :: PF : FG$, and by sim. Δ s, $BD : BP :: FG : PF$.

Wherefore it is evident, that the rectangle $BD \cdot CE$ is equal to the given rectangle $PB \cdot PI$. Q. E. D.

The same, answered by Mr. Richard Nicholson, Liverpool.

GEOMETRICAL ANALYSIS. Fig. 454. Pl. 24. Join the points P, C, and P, B; draw $BF \parallel$ to CP, and PG to AC, the former to meet EP in F, and the latter BF in G. Then (Euc. 4. VI.) $EC : CP :: GP : GF$, theref. $EC \cdot GF = CP \cdot GP$ a given rectangle, and $EC \cdot BD$ is given, by hyp.; theref. (Euc. Dat. 65.) the ratio of GF to BD is given. Make the $\angle GPH = FPD$ the sup. of the given one EPD, and $PHI = FGP$; let H be in AB, and I in PD; then (Euc. 4. VI.) $PG : PH :: GF : HI$, but PG and PH are given, theref. the ratio of GF to HI is given, and, it has been shewn, that the ratio of GF to BD is given; theref. (Data 9.) the ratio of BD to HI is given. Draw BK and HL \parallel to PI, BM and DN \parallel to HI, and LP and OQ \parallel to AB; let DN meet HL in O, and let QO meet BM in S. Then (Simp. Geo. 21. IV.) $RM : OS :: RK : OQ = HB$, theref. $OS \cdot RK = RM \cdot BH$. And, by reason of parallel lines, $HI = DO = BS : BD = OS :: BM : MR$, but the ratio of HI to BD has been shewn to be given, theref. the ratio of BM to MR is given, and BM is a given line, theref. MR is given, and BH being given, the rectangle $MR \cdot BH$ will be given; therfore, its equal, the rectangle $OS \cdot RK$ is given, that is, the rectangle $KP \cdot KR$ is given (because $OS = BD = KP$). Hence the rectangle $KP \cdot KR$, and PR being given, the point K may be found by Euc. 27, 28. VI.; consequently the position of PD becomes known.

CONSTRUCTION. Draw PC, PB, draw also BG, PG, parallel to CP and AC respectively, and take any two lines m and n in the given ratio of the rect. $CP \cdot PG$ to $EC \cdot BD$; make the $\angle GPH =$ the sup. of EPD, and $PHI =$ the sup. of BGP; take the line p a 4th proportional to PG, PH and m , draw PM \parallel to AB and BM \parallel to HI; and take MR a 4th proportional to p , n , and BM. Divide PR in K so that the rect. $RK \cdot KP$ may be $= MR \cdot BH$, join KB and parallel thereto draw PD, and draw PE to make an \angle with PD equal to the given one, and the thing is done.

Messrs. *Cunliffe* and *Swale* observe, from Mr. *Wallace's* Analysis to Quest. 152, that this prob. is reducible to what follows.

Fig. 351. Pl. XXI. The lines GK and CE are given by position, and C and G are given points therein; it is required to draw a line PKE through another given point P, to cut the former lines in K and E, so that the rectangle GK : CE may be equal to a given space.

This prob. may be seen done in all its various cases in a small work of Dr. HALLEY's, printed at Oxford in the year 1706, entitled, *Apollonii Pergæi de sectione spatii libri duo restituti*.

XXVI. Or, PRIZE QUESTION 182, answered by
Arithmeticus, of Newcastle upon Tyne.

Let ABCD (fig. 455. pl. 24) represent a section of the vessel, E the place of the eye, Eg an incident ray refracted at g into gt, and let Eg be continued to b. Suppose cVd to represent the bottom of the vessel as it appears to the eye at E; then the point t will be seen at s, F at V, &c.—Draw sR || to CD and put EG = a, GF = b, ER = x and sR = y; then will Es = $\sqrt{(x^2 + y^2)}$, and the sine of the \angle of incidence sER = $y \div \sqrt{(x^2 + y^2)}$: but by sim. Δ s $x : y :: a : ay \div x = Gg = FH$, and $y - ay \div x = Ht$, whence the sine of the \angle of refraction tgH = $(xy - ay)$

$\div \sqrt{(b^2x^2 + xy - ay)^2}$, which is known to be in the constant ratio of 3 to 4 to the sine of the \angle of incidence, therof. $3y \div \sqrt{(x^2 + y^2)} = (4xy - 4ay) \div \sqrt{(b^2x^2 + xy - ay)^2}$, which reduced gives $y^2 = 9b^2x^2 \div 7(x - a)^2 - 16x^2 \div 7$. When $y = 0$, then $x = a + \frac{1}{2}b$, or $FV = \frac{1}{2}FG$, and when $y = CF$ or 1, then $x = 1.1501574$ nearly.

The fluxion of the capacity ABcd (putting $p = 8.14159$, &c.)

is $(16x^2x - 9b^2x^2 \cdot \frac{x-a}{x-a})^{-\frac{1}{2}} \times \frac{1}{7}p$, the correct fluent of

which is $\frac{16p}{21} \left(x^2 - \frac{4x + 3b^2}{4} \right)^{\frac{1}{2}} + \frac{9pb^2}{7} \left(\frac{9b^2 - 16a^2}{12b} - \frac{x^2 - 2ax}{x - a} \right)$

$+ \frac{18ab^2p}{7} \times \text{hyp. log. of } \frac{3b}{4x - 4a}$, which, when $a = \frac{1}{2}$, $b = 1$, and

$x = 1.1501574$, is $= .0490014 p$, and therefore the capacity of the vessel $ABcd$, as it appears from E , is $.6991588 p$, which taken from the whole capacity leaves $.3008412 p = .9451197$ cubic feet, the difference required.

Mr. Gregory, of Cambridge, gave a solution, nearly similar to the above, by Arithmeticus, (making the required difference 1636.24 inches, or 5.8022 ale gallons,) at the end of which he added the following observations:

1. From the investigation of the nature of the curve given above, it will appear, that it bears a near affinity to the *superior conchoid* of *Nicomedes*, the surface of the water being an asymptote to the apparent bottom, and corresponding to the *directrix* of the conchoid.

2. As the eye approaches the surface of the water, the extremities of the apparent bottom will have a greater curvature; approximating more and more to the nature of an ellipse as the eye approaches the water.

3. In the above solution I have supposed the lines of incidence and refraction to be as 4 to 3, for the sake of simplifying the calculation; this, though not quite accurate, is so near the truth, as not to produce any material error. I have also supposed that the point C , in the circumference of the base, is refracted so as to appear at c exactly above it: this (though nearly accurate in the present instance) will not be the case when AG and GF are very large in proportion to EG . A rule for finding the situation of the refracted point c , in most cases, may be found in Dr. BARROW's *Optics*. Lect. 16.

The same, answered by Mr. James Cunliffe, Bolton.

Fig. 455, Pl. 24. Let $ABCD$ represent a section of the cylinder, by a plane passing through the axis FG , E the place of the eye therein: also let the curve cVd represent the image of the line CD , as it appears in the water to the eye at E . Draw $tn \perp$ to CD , intersecting the curve in s , draw also Es meeting AB in g and join tg ; then will s be the apparent place of the point t , the $\angle GEg = Esn$ will be the \angle of incidence, and the $\angle ntg$ that of refraction of the ray Es : see *Emerson's Optics*, Cor. 6, Prop. 9, B. III. and Prop. 29 of the same Book.—Now in order to find the equation of the curve, which is the *locus* of all the points s , upon FE demit the \perp sR : put $EG = b$, $GV = d$, $FV = c$, $VR = x$, and $Rs = y$, then $ER = b + d - x$, and by Euc. 47,

$$1. Es = \sqrt{ER^2 + Rs^2} = \sqrt{(b + d - x)^2 + y^2}. \text{ By reason of}$$

$$\text{the parallels } Gg, Rs, ER : Es :: GR : gs = (d - x \div b + d - x)$$

| 2 1 2 ✓

$\sqrt{b+d-x}^2 + y^2$, and because of the sim. $\Delta s ERs$ and mg, ER
 $: Rs :: ns = GR : ng = y(d-x) \div (b+d-x)$, whence, and
 by Euc. 47. I. tg

$$= \sqrt{(tn)^2 + (ng)^2} = \sqrt{(d+e)^2 + y^2(d-x)^2 \div (b+d-x)^2}$$

$$= \sqrt{(d+e)^2(b+d-x)^2 + y^2(d-x)^2 \div (b+d-x)}.$$

Moreover, by the principles of trigonometry and optics, tg and gs
 are in the constant ratio of the sine of the angle of incidence to that
 of refraction, that is, as $d+e$ to d , or

$$\sqrt{(d+e)^2(b+d-x)^2 + y^2(d-x)^2 \div (b+d-x)} : (d-x)$$

$\sqrt{(b+d-x)^2 + y^2 \div (b+d-x)} :: d+e : d$; by equat-
 ing the product of the means and extremes, and proper reduction,
 we have $y^2 = (d+e)^2(2dx-x^2)(b+d-x)^2 \div (2de+e^2)$
 $(d-x)^2$, being an equation expressing the nature of the curve
 \sqrt{Vd} , when the abscissa and its corresponding perpendicular ordinate
 begin together. Now $y^2 px = (d+e)^2(2dx-x^2)(b+d-x)^2$
 $(2de+e^2)^{-1}(d-x)^{-2} \times px$ will express the fluxion
 of the solid generated by the revolution of the space VsR about
 the axis FE . Put $v = d-x$, and the preceding expression be-
 comes $(d+e)^2(d^2-v^2)(b+v)^2(2de+e^2)^{-1} \times pv^{-2}$
 $\times -\dot{v}$. But $(d^2-v^2)(b+v)^2 \times v^{-2} \times -\dot{v}$ is $= -$
 $d^2\dot{v}v^{-2} - 2bd\dot{v}v^{-1}\dot{v} - d^2\dot{v} + (b+v)^2\dot{v}$, the
 fluents of which are $d^2b^2v^{-1} + \frac{1}{3}(b+v)^3 - d^2v - 2bd^2$
 \times hy. log. of v ; therefore, the expression for the content of the
 solid generated by the revolution of the space VsR about the axis
 FE will be $p(d+e)^2(2de+e^2)^{-1} \times (d^2b^2v^{-1} + \frac{1}{3}$
 $(b+v)^3 - d^2v - 2bd^2 \times$ h. l. of $v) = p(d+e)^2(2de+e^2)^{-1}$
 $\times (d^2b^2(d-x)^{-1} + \frac{1}{3}(b+d-x)^3 - d^2(d-x) -$
 $2bd^2 \times$ h. l. of $(d-x)$, by restoring the value of v , (p being $=$
 $3.14159265\ldots$). But the whole expression ought to vanish when
 $x=0$, therefore the correct expression for the content of the said
 solid will be $p(d+e)(2de+e^2)^{-1} \times (2bd^2 \times$ h. l. of d
 $(d-x)^{-1} + b^2d^2(d-x)^{-1} + \frac{1}{3}(b+d-x)^3 - b^2d$
 $-\frac{1}{3}(b+d)^3)$. The preceding part of the solution is general,
 and will answer *mutatis mutandis* for any other medium besides
 water.

When

When s coincides with c , then $y = GA = GF = d + e$ by the question, therefore $y^2 = (d + e)^2 = (d + e)^2 (2dx - x^2)$
 $(b + d - x)^2 (2de + e^2)^{-1} (d - x)^{-2}$, whence $(2de + e^2)$
 $(d - x)^2 = (2dx - x^2) (b + d - x)^2$; in the present case, taking the ratio of the line of the angle of incidence to that of refraction, in water, as 4 to 3, we have $d = 9$, $e = 3$ and $b = 6$; wherefore, the preceding equation becomes $63 (9 - x)^2 = (18x - x^2) (15 - x)^2$, or $x^4 - 48x^3 + 828x^2 - 5184x + 5103 = 0$, from whence $x = 1.1981314$. By means of this value of x and the foregoing general expression for the solidity, the content of the part generated by the revolution of the space cVR' about the axis FE will be found to be 266.0221559 cubic inches; and the content of the cylindrical part $cdDC$ is easily found to be 1899.1899 cubic inches, whence their difference. 1633.1677441 cubic inches is the content of the part $cVdDC$, comprehended between the apparent and real bottoms of the vessel, which is, therefore, the quantity of water required.

OBSERVATIONS. 1. The line AB produced is an asymptote to the continuation of the curve cVd , which has two infinite branches or legs. For, in the equation of the curve, when $x = d$, y becomes infinite or an asymptote. Hence Mr. *Emerson* appears to have inadvertently been guilty of an error in Prop. 29. B. 3rd. of his *Optics*, for he there says, that the curve cVd , if continued, would meet the surface of the fluid, but this is impossible, when the line CD is \parallel to the surface.

$$2. \text{ From the equation of the curve } y \text{ is } = \frac{b + d - x}{d - x} \times \frac{d + e}{\sqrt{2de + e^2}} \\ \times \sqrt{2dx - x^2} = \frac{d + e}{\sqrt{2de + e^2}} \times \frac{b + v}{v} \times \sqrt{d^2 - v^2}.$$

Now, if a semi-ellipsis be described, having G for its centre, GV for its semi-conjugate, and $\frac{(d + e) \times GV}{\sqrt{2de + e^2}}$ for its semi-trans-

verse axis, and an ordinate be drawn thereto at right angles to GV , and the point where the ordinate meets the ellipsis and the centre G be joined; then if a right line be drawn through E , parallel to the latter line, it will meet the said ordinate in a point which will be in the curve cVd .

3. By comparing the equation of the curve, $y = \frac{d+e}{\sqrt{2de+e^2}} \times$

$\frac{b+v}{v} \times \sqrt{d^2-v^2}$, with that in *Cor. to Prop. IV. Emerson on the*

conchoid, it will appear that the curve is really a species of the conchoid. Now, if instead of the quadrant of a circle, as in the *III. Prop. of Emerson*, we imagine the quadrant of the ellipsis, mentioned in the last observation, to be described, the property of the conchoid, stated in that proposition, will equally appertain to the curve cVd .

4. The curve may thus be described by continued motion. If an ellipsis whose principal semi-axis is $\frac{(d+e)d}{\sqrt{2de+e^2}}$ and its semi-

conjugate d , be moved along its plane always coinciding with the plane passing through the line AB and the point E; and its transverse axis at the same time coinciding with the line AB: then if a right line be drawn through the point E and the centre of the ellipsis, the intersection of this line with its periphery, on the side of AB, which is farthest from E, will generate the curve cVd .

For let $psvq$ (fig. 456, pl. 24.) represent this ellipsis as its transverse axis moves along the line AB: through the centre g draw the line Egs to meet the periphery of the ellipsis in s , draw also sn and sR perpendicular to pq and VE respectively, the latter intersecting the semi-conjugate axis vg in r . Put the semi-transverse

$gp = \frac{(d+e)d}{\sqrt{2de+e^2}}$, the semi-conjugate $gv = GV = d$, $EG = b$,

$VR = vr = x$, and $sR = y$; then $ER = b + d - x$, and $GR = gr = d - x$. By the well known property of the ellipsis, $rs =$

$\frac{(d+e)\sqrt{2dx-x^2}}{\sqrt{2de+e^2}}$, and by reason of the similar Δsrg and

sRE , $sr : rg :: sR : RE$, whence $sR = \frac{sr \times RE}{rg}$, this, being

expressed algebraically, is $y = \frac{d+e}{\sqrt{2de+e^2}} \times \frac{b+d-x}{d-x} \times \sqrt{2dx}$

$\sqrt{2dx - x^2}$ which is the equation derived in the question. If, instead of an ellipsis, we imagine a circle to be moved along, in the same manner, the curve generated thereby will be the conchoid of Nicomedes, as is very well known, and the circle is denominated the *generating circle*, wherefore in the curve Vs the above-mentioned ellipsis may not unaptly be called the *generating ellipsis*.

The same, answered by Mr. John Lowry, Birmingham.

Fig. 455, Pl. 24. Let ABCD represent a section of the cylindrical vessel, made by a plane passing through the axis EF, E the place of the eye, and the curve cVd the form of the line CD, as it appears to the eye at E. Let Eg be any ray from E falling on the surface of the water at g, produce Eg to meet the curve at s, and draw nst \perp to CD; join tg, and draw sR, cd parallel to CD, to meet the axis EF in R, R' respectively. Then, by the principles of Optics, the ray Eg instead of proceeding in the straight direction Egs, will, by refraction, be bent into the oblique direction gt, so that the point t, as seen from E, will appear at s, and gs will be to gt in the constant ratio of 3 to 4. In the same manner, the centre of the bottom F, will be elevated to V, and EV will be to EF as 3 to 4: hence $GV = .75$ and $VF = .25$. Now put $EG = a = .5$, $GV = .75$, $EV = m = a + b = 1.25$, $VF = c = .25$, $RV = x$ and $Rs = y$, then by sim. Δ s and Euc. I. 47, $ER^2 : Es^2 = ER^2 + Rs^2 :: GR^2 : (gs)^2 = (b - x)^2$

$+ y^2 (b - x)^2 (m - x)^{-2}$, hence $(ng)^2 = (gs)^2 - (ns)^2 (GR^2)$
 $= y^2 (b - x)^2 (m - x)^{-2}$, and $(gt)^2 = (tn)^2 + (ng)^2 = (b + c)^2$
 $+ y^2 (b - x)^2 (m - x)^{-2}$; $\therefore (gs)^2$ or $(b - x)^2 + y^2 (b - x)^2$
 $(m - x)^{-2}$; $(gt)^2$ or $(b + c)^2 + y^2 (b - x)^2 (m - x)^{-2} ::$
 GV^2 or $b^2 : GF^2$ or $(b + c)^2$, and by multiplying means and extremes, we obtain, after reduction, $y^2 = (b + c)^2 (b^2 - \overline{b - x})^2$
 $(m - x)^2 (2bc + c^2)^{-1} (b - x)^{-2} = n (b^2 - \overline{b - x})^2 \times$
 $(m - x)^2 (b - x)^{-2}$ an equation expressing the nature of the

curve cVd, n being $= (b + c)^2 (2bc + c^2)^{-1}$.

Now, when s coincides with c, y is $=$ to CF $= 1$, by the question, and, in that case, the above equation, by reduction, becomes

comes $x^4 - (2m + 2b)x^3 + (4m^2 + m^2 + \frac{1}{2}n)x^2 - (2bm^2 + 2b \div n) - b^2 \div n = 0$, in which the value of x is $\cdot 0998443$.

Again, to find the content of the solid, generated by the revolution of the space VsR about the axis VR , put $p = 3 \cdot 1416$,

then $py^3x = pn(b^2 - b - x)^2(m - x)^2(b - x)^{-2}x$ is the fluxion of the said solid. And to find the fluent, it is evident that

$(b^2 - b - x)^2(m - x)^2(b - x)^{-2}x$, may be resolved into $(b - x)^{-2}$

$\times a^2b^2x, (b - x)^{-1} \times 2ab^2x, b^2x$, and $-(m - x)^2x$, the

fluents of which (found by the common rules) are $a^2b^2(b - x)^{-1}$, $-2ab^2 \times \text{hy. log. of } (b - x)$, b^2x , and $\frac{1}{3}(m - x)^3$ respectively; therefore the fluent of py^3x , or the expression for the content of the solid generated by the revolution of the space VsR about VR

is $= pn \times (a^2b^2(b - x)^{-1} - 2ab^2 \times \text{hy. log. of } (b - x) + b^2x + \frac{1}{3}(m - x)^3)$. But when x is $= 0$, this expression becomes

$pn \times (a^2b^2(b - x)^{-1} - a^2b + 2ab^2 \times \text{hy. log. of } (b \div b - x) + b^2x + \frac{1}{3}(m - x)^3 - \frac{1}{3}m^3)$, which, when x is $= VR' = \cdot 0998443$ (as found above), becomes $\cdot 158948$ cubic feet, the content of the solid, generated by the revolution of the space VcR' about VR' . And the content of the cylinder $CedD$ is easily found $= 1 \cdot 099068$; consequently their difference, or $\cdot 94514$ cubic feet is the quantity of water required.

When x is $= b$, the value of y in the above equation of the curve is infinite, which shews, that AB produced is an asymptote to the continuation of the curve. This is also evident from another consideration, for if the water be extended on all sides indefinitely, it is plain that, every part of the curve will fall below the surface of the water, when that surface is considered as a plane parallel to the horizontal plane CD . The curve will therefore have two infinite legs and in form will much resemble the conchoid of Nicomedes.

The same, answered by Mr. John Surtees, Bishop Wearmouth, Sunderland.

Let Fig. 455, Pl. 24, represent a vertical section of the vessel, E the situation of the eye, $EG = \frac{1}{2}$, $AG = GF = 1$ foot, $AR = nG$

$= nG = y$, $GR = x$, $c = 3.1416$, and $m = 2.30258509$ hyp. log. of 10. By sim. $\Delta s ER (\frac{1}{4} + x) : sR (y) :: sn (x) : ng =$

$2xy \div (1 + 2x)$, and by Euc. I. 47, $\sqrt{nl^2 + ng^2} = gt =$

$\sqrt{1 + x^2y^2 \div (\frac{1}{4} + x + x^2)}$ and $\sqrt{nl^2 + ng^2} = sg =$

$\sqrt{x^2 + x^2y^2 \div (\frac{1}{4} + x + x^2)}$, which, by the nature of refraction, are to one another as 4 to 3; hence $y^2 = (1 \div 28x^2) \times (9 \times 36x + 20x^2 - 64x^3 - 64x^4)$, and when $y = 0$, $x = \frac{3}{4}$, and when $y = 1$, $x = .65016$.—Now, the fluxion of the solid, generated by the revolution of the area R'sV about R'V, =

$cy^2 \times -x$, and the fluent $= (c \div 28) \times (9x^{-1} - 36m \times \log. x - 20x + 32x^2 + (64 \div 3)x^3)$, and, when duly corrected,

the fluent is $= (c \div 28) \times (9x^{-1} - 24 + 36m \times \log. (3 \div 4x) - 20x + 32x^2 + (64 \div 3)x^3)$, which, when $x = .65016$, becomes $= .15416$; therefore $(R'c)^2 \times R'F \times c = .15416 = 1 \times .84984 \times 3.1416 = .15416 = .945$ solid feet $= 1032.96$ cubic inches = the solid generated by the area CcVF about VF. W. W. R.

True solutions were also received from Messrs. Hill, Nicholson, Thorpe, the proposer, and Robert Wallace. Other gentlemen attempted solutions, but without success.

The Prize Medal is decided in favour of Mr. James Cunliffe, who will please to send for it to Mr. Glendinning's.

ARTICLE XLIV.

MATHEMATICAL QUESTIONS.

(To be answered in Number XII.)

I. QUESTION 211, by Mr. John Croudace, *Lanchester School, Durham.*

ON the first day of May, 1800, the sum of the sun's declination, altitude at six, and amplitude was found to be equal to the latitude of the place. Query where?

II. QUESTION 212, by Mr. Francis, *Hampstead.*

There is a certain hill from whose summit may be seen four separate objects, A, B, C, D, distant from each other 23, 20, 30, and 18·909 miles respectively. Now admitting all those objects to be in the horizon, or utmost extremity that the eye can reach, from that elevation, what is the perpendicular height of the hill, considering the earth as a perfect sphere 25000 miles in circumference.

III. QUESTION 213, by Mr. John Blackwell, *Hungerford, Berks.*

There are two circular segments, of equal altitude, touching each others vertex, viz. the convex vertex of the less touching the concave vertex of the greater segment, the length of whose arcs are 50 and 60 yards respectively. Now, it is observed, that the area of the less segment is to the area of its greatest inscribed circle, in the ratio of 5 to 2. Whence it is required to find the area included between the arcs of those circles from which the segments were cut.

IV. QUESTION 214, by Mr. Wm. Marrat, *Boston.*

Three sailors, having found an ingot of gold, in form of a cone, the length of which is 4 and the diameter of its base 6 inches, are determined to have it divided equally among them, by sections parallel to one side: but finding the operation to be rather difficult, request some of the learned contributors to the Repository will favour them with a method of solution.

V. QUESTION 215, by Mr. Thomas Squire, *Baldock.*

One night I made observation on an eclipse of the moon, but clouds interposing I could not see the beginning; however at 11^h. 49^m. 48^s. it cleared up, when she was exactly 6 digits eclipsed on her lower limb, at the same time I measured the *distance* between the extremities of the illuminated cusps, and found it to be 30' 15": the air continuing bright, at the time of the greatest obscuration, I found the digits eclipsed to be 7d 27m. 36s.—soon after clouds appearing put a stop to all farther observation. By calculation I found the moon's semidiameter 15' 24", and her horary motion, from the sun, on the relative orbit, 29' 45". From these data, the beginning, middle, and end are required.

VI. QUESTION 216, by Mr. W. Burdon, *Acaster-Mallis.*

Given the base, the diameter of the circumscribing circle, and the difference between the perpendicular and the radius of the inscribed circle, to construct the plane Δ .

VII. QUESTION 217, by Mr. Newton Bosworth, *Cambridge.*

There is a lever, equally thick throughout, the length of which is 10 feet and weight 40lbs averdupoise; from one end of which is suspended a globe of lead of 6 inches diameter; it is required to determine where the fulcrum must be placed so that the whole may remain in equilibrio.

VIII. QUESTION 218, by Mr. Jonathan Barr, *Student in Mr. R. Wallace's Academy, St. John's-Lane, West Smithfield, London.*

Given the ratio of the base to the diameter of the circumscribing circle, the ratio of the difference of the segments of the base to the perpendicular, and the radius of the inscribed circle, to construct the plane Δ .

IX. QUESTION 219, by Mr. Thomas Simpson Evans.

Required a simple and accurate expression to reduce the apparent latitude to that at the centre?

X. QUESTION 220, by Mr. Wm. Peacock, *Birmingham.*

In a given ΔABC it is required to inscribe another EDF , whose angular points E, D, F shall fall on the sides AB, BC, AC respectively, so that the two sides FE, DE shall pass through two given points I, H , and moreover, that the third side FD shall cut off from the given Δ another FCD of a given magnitude; or that the sum of CF, CD shall be equal to a given line; and shew the limits in both cases.

XI. QUESTION 221, by Mr. John Byerley, *Stockton-upon-Tees.*

At what point in a plane passing through the centres of the sun, earth, and moon, must a particle of matter be placed so as to be equally attracted by these bodies, their distances and magnitudes being given?

XII. QUESTION 222, by Tyro Philo Mathematicus, of Hull.

A certain water pool, the area of which is 10 acres, has two small outlets through two similar and equal rectangular apertures, each 3 feet wide parallel to the horizon and 4 feet high perpendicular thereto:—Now admitting the surface of the water to be level with the top of the apertures when it begins to issue out through them; I would know in what time the said surface will be on a level with the bottoms of the said apertures?

XIII. QUESTION 223, by E. R.

Supposing a solid of a given base and altitude, to be generated by the rotation of a given equilateral hyperbola, round one of its asymptotes, as an axis;—It is required to determine the length of a spiral, making a given number of turns round the same, from the base to the top, in such a manner, that if ordinates were drawn from the beginning of each turn to the asymptote, they would divide the axis into the same number of equal parts?

XIV. QUESTION 224, by Mr. Olinthus Gregory, Bookseller, Cambridge.

The sides of a \triangle are 30, 26 and 24. Now if this \triangle be placed on fulcrums at the angular points, and a weight of 100 lbs be laid on the centre of the inscribed circle, what part of this weight will be supported by each of the three fulcrums; and how far from the angular points must the weight be laid so that each fulcrum may support an equal part of it?

XV. QUESTION 225, by Mr. James Whalley.

Required the sum of n terms of the series $\frac{1}{1+x} + \frac{2x}{1+x^2} + \frac{4x^3}{1+x^4} + \frac{8x^5}{1+x^8} + \&c.$?

XVI. QUESTION 226, by Mr. Gregory.

There are two wheels (the weights of which are supposed to be condensed into their circumferences;) one of them whose radius is 2 feet, weighs 50lbs, and from a cord, coiled round an axle of $\frac{1}{4}$ of a foot radius, a weight of 5lbs is suspended: the other wheel,

in radius $1\frac{1}{2}$ feet, weighs 40 lbs. and has a weight of 5 lbs. suspended in like manner, from an equal axle. Suppose these two wheels to move freely on their respective axles, it is required to shew, how long they have been in motion, when the weight, affixed to one axle, has descended 50 feet farther than that suspended at the other?

XVII. QUESTION 227, by Mr. James Cunliffe, Bolton-le-Moors, Lancashire.

Required the least dimensions of a triangular plot of land, such that if perpendiculars be drawn to the sides, from their opposite angles, the lengths of the said perpendiculars and the segments of the sides, made thereby, may be expressed in whole chains?

XVIII. QUESTION 228, by E. R.

To describe a circle through two given points to cut an indefinite right line, given by position, and which is situated between the two given points, so that if two right lines be drawn from one of the given points to the points of intersection of the right line and circle, they may include a given angle.

XIX. QUESTION 229, by Mr. Robert Wallace, London.

Suppose a corpuscle of matter to be situated in the axis produced of a given paraboloid. It is required to determine the whole force of attraction of the paraboloid upon the corpuscle: supposing each particle to exert a force as the n th power of its distance.

XX. QUESTION 230, by Mr. James Cunliffe, Bolton.

To find two rational fractions such that their sum, sum of their squares, and sum of their cubes shall all be rational square numbers; and moreover, either of them being added to the square of the other shall make a rational square number.

XXI. QUESTION 231, by Mr. Smith, of Liverpool.

Given one side, an adjacent angle, and the area to construct the trapezium, when the side, opposite to the given side, produced passes through a given point in the given side produced, and the angle, opposite to the given angle, lies in a straight line given by position.

XXII. QUESTION 232, *by Mr. Louis Hill.*

Required a new, concise and elementary demonstration of the rule, in common use, for finding the area of a trapezium inscribed in a circle, when the four sides are given.

XXIII. QUESTION 233, *by Mr. John Johnson, Birmingham.*

In any plane triangle, if a diameter of the inscribed circle be drawn perpendicular to the base, to meet the line bisecting the base at P, and Q and R be the points of contact of the circle with the other two sides of the Δ ; the points Q, P, R lie in a straight line. Required the demonstration?

XXIV. QUESTION 234, *by Geometricus.*

In the figure to Prop. 47, Book I. of all the editions of Euclid's Elements which I have seen, the straight lines joining the points F, C; B, K, appear to intersect the perpendicular AL in the same point. It is required to demonstrate whether they really do meet in the same point or not. See the figure in Dr. Simson's or Professor Playfair's Editions.

XXV. QUESTION 235, *by Mr. Louis Hill.*

Though a given point within or without a given circle, to draw a right line to intersect the circle so that if perpendiculars be drawn from the points of intersection to that diameter of the circle which passes through the given point, the sum, difference or rectangle of the said perpendiculars may be of a given magnitude.

XXVI. QUESTION 236, *by Mr. Lowry, Birmingham.*

To determine a point, such, that if three straight lines be drawn from it to meet three other straight lines, given by position, in given angles; the sum of the lines so drawn may be equal to a given line; and moreover, that the rectangle contained by two of them may be equal to the rectangle contained by the third and a given line.

X VII. QUESTION 237, *by the Rev. L. Evans.*

Mr. Worldly-wise-man lends a tradesman, newly set up in bu-
siness

finest, 1000l. to bear compound interest at 5 per cent. per annum, till the same be liquidated, by quarterly payments, thus, 1l. at the end of the 1st quarter, 2l. at the end of the 2nd quarter, 3l. at the end of the 3d quarter, &c. It is required to find, by a clear and general investigation, how long the tradesman will be in acquitting himself of this obligation?

Note. A similar question, in effect, has been proposed before, but the solution given to it is not satisfactory.

XXVIII. QUESTION 238, by Mr. Geo. Sanderfon.

If there be three arches or angles δ , η , ϕ , where δ is given,

the expression $\frac{\sin(\delta - \eta - \phi)}{\sin \phi} \times \sqrt{\cos \eta}$ will be the greatest

possible when the $\tan(\delta - \eta) = \frac{1}{2} \tan \eta \times \frac{2 - \tan \eta \times \tan \phi}{\frac{1}{2} - \tan \eta \times \tan \phi}$;

the relation between the angle η and ϕ being expressed by the equation, $\cos \eta = (a^2 \div 2b^2) \times \tan \phi^2$.—See page 222, Watson's Translation of Euler's construction and properties of vessels. Required a demonstration?

XXIX. QUESTION 239, by Mr. Lowry.

Let AB, AC be two straight lines given by position, B and C given points in these lines, and P a given point without them. It is required to draw two straight lines PD, PE, meeting the given lines in D and E, so that the angle DPE may be of a given magnitude, and the sum or difference of the rectangles BD \times M, CE \times N may be equal to a given space; M and N being given lines.

XXX PRIZE QUESTION 240, by Mr. Lowry.

In a circle given in magnitude and position, it is required to inscribe a trapezium such, that, if its opposite sides be produced, they may intersect each other in two given points, and that one of its diagonals may also pass through a given point.

ARTICLE XLV.

SEVEN PROPOSITIONS FROM LAWSON.

(To be answered in No. XII.)

PROP. XL.

IF there be two triangles ABC, DEF, which have one angle A in one equal to one angle D in the other, and another angle B in the first equal to the sum of the angles D and E in the second; then shall the sides AC, BC, DE, EF be proportional.

PROP. XLI.

The square of the line bisecting the vertical angle of any triangle is a mean proportional between the difference of the squares of each side including that angle, and the square of the adjacent segment of the base made thereby.

PROP. XLII.

If from the same point two tangents be drawn to a circle, and a line be drawn joining the points of contact, and another line to be intercepted between the tangents cut the foregoing which joins the points of contact, so as to be bisected in the point of intersection; then I say that the part of that line which is a chord of the circle will also be bisected by the same point.

And, conversely, if the chord cutting the line joining the points of contact be bisected by the point of intersection, then the continuation of the same to meet the tangents will also be bisected by the same point.

PROP. XLIII.

If from one of the equal angles of an isosceles triangle a perpendicular be drawn to the opposite side; then I say that the rect-
angle

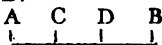
angle contained under that side and its segment intercepted by the perpendicular and the base is equal to half the square of the base.

PROP. XLIV.

If in a line AB two points C and D be taken, then I say that $(AB + AD) \times BC + BC^2 = 2ABC + BCD$.

And moreover that

$$(AB + AD) \times CD + CD^2 = 2ADC + BCD.$$



PROP. XLV.

If from the vertical angle of any triangle two lines be drawn to make equal angles with the sides containing it, and to cut the base; then I say that the square of one side is to the square of the other side as the rectangle under the segments of the base contiguous to the first side is to the rectangle under the segments contiguous to the other side.

PROP. XLVI.

If in AB the diameter of a semicircle any point C be taken, and from thence any line as CD drawn to meet the circumference in D, and a perpendicular DE be demitted; then I say that the square of the line AC is equal to the square of the line CD together with the rectangle under the sum of the distances of C from A and C from B and the line AE, when C is taken in the diameter AB produced; but equal to the square of CD together with the rectangle under the difference of the distances of C from A and C from B and the same line AE, when C is taken in the diameter itself.

ARTICLE XLVI.

DAWSON ON THE DISTANCE OF THE SUN.

(Continued from page 234.)

PROP. IV. Fig. 457, Pl. 25.

LET ADE be a circle described round the centre T, and suppose that a body describes it by a force, tending to T, which is inversely as the square of the distance from T; if another force, which is as its distance, act upon the body directly from the centre, and, by the compound force, make it describe the curve ABC; it is required, from having the proportion of these two forces given, to determine the motion of the apses, and *e contra*, from having the motion of the apses given, to determine the proportion of the two forces.

Put $TA = r = 1$, $TB = x$, $GB = y$, $AB = z$, $Cd = \dot{x} \pm \dot{y}$, $BC = \dot{z}$, and $BD = \dot{u}$. Let the force of the body T, at the distance TA, be represented by r or 1 , and the disturbing force by n . Then, at the distance TB or x , the two forces that act upon

the body, moving at B, will become $\frac{r^3}{x^2}$ and $\frac{nx}{r}$. The

velocity of the body in the circle AGD, is the same as would be acquired by falling from a height above A $= AT$ or x , which therefore is easily found to be $= r$. [See prop. 3, tract 3, of *Traacts Physical and Mathematical*.] It is likewise commonly known, that the central force, multiplied into \dot{x} , is equal the velocity into the fluxion of the velocity. Let the velocity, at B, be put $= v$, then, we have this equation,

tion, $\frac{r^3}{x^2} - \frac{nx}{r} \times \dot{x} = -v\dot{v}$, and, taking the fluents,

$$-\frac{r^3}{x} - \frac{nx^2}{2r} = -\frac{v^2}{2}. \quad \text{But, at A, the equation}$$

$$\text{becomes } -\frac{r^3}{r} - \frac{nr^2}{2r} = -\frac{r^2}{2}; \quad \text{therefore, by}$$

correction and reduction, we have $\frac{r^2}{x} \times \frac{1}{2r-x} +$

$$\frac{n}{r} \times \frac{1}{x^2-r^2} = v^2. \quad \text{Suppose AN to be described,}$$

in the same time as BC; then, from the triangle ATN, being equal the triangle BTC, we gain the following proportions, viz.

$$\begin{array}{llll} \text{and} & \text{---} & \text{---} & \text{AN}^2 : \text{BC}^2 :: r^2 : v^2, \\ & & & \text{Bd}^2 : \text{AN}^2 :: r^2 : x^2. \\ \text{therefore, ex aequo} & & & \text{Bd}^2 : \text{BC}^2 :: r^4 : x^2 v^2, \\ \text{that is,} & \text{---} & \text{---} & \dot{u}^2 : \dot{x}^2 :: r^4 : x^2 v^2. \\ \text{By division} & \text{---} & \text{---} & \dot{u}^2 : \dot{x}^2 :: r^4 : x^2 v^2 - r^4, \\ \text{hence} & \text{---} & \text{---} & \dot{u} : \dot{x} :: r^2 : \sqrt{x^2 v^2 - r^4}, \end{array}$$

$$\text{consequently } \dot{u} = \frac{r^2 \dot{x}}{\sqrt{v^2 x^2 - r^4}}. \quad \text{In the value of } v^2,$$

found above, substitute $r+y$ for x , and, neglecting all the terms where y is found above the second power, we shall have, when reduced, \dot{u}

=

$$= \frac{r^2 \dot{y}}{\sqrt{2nr^2 y - r^2 - 5nr} \times y^2} = \frac{r^{\frac{3}{2}} y - \frac{1}{2} \dot{y}}{\sqrt{2nr - r - 5n} \times y}.$$

Therefore $\overline{r + y} \times \dot{u} = \frac{r^{\frac{5}{2}} y - \frac{1}{2} \dot{y}}{\sqrt{2nr - r - 5n} \times y} +$

$$\frac{r^{\frac{3}{2}} y^{\frac{1}{2}} \dot{y}}{\sqrt{2nr - r - 5n} \times y} = \text{twice the area of the fluxion-}$$

ary triangle BTC. When the body comes to the apogee, $\dot{u} = \dot{z}$ or $\dot{y} = 0 = \dot{x}$, that is, the denomi-

nator $\sqrt{2nr - r - 5n} \times y = 0$; therefore, sup-

posing $r = 1$, y will be found $= \frac{2n}{1 - 5n}$. Taking

the fluent of the above fluxion, at that time, and still supposing $r = 1$, we have twice the area described, while the body passes from perigee to apogee, or the area described from apogee to apogee

$$= \frac{c}{\sqrt{1 - 5n}} + \frac{nc}{1 - 5n^{\frac{3}{2}}}, \quad (c \text{ being } = 3.14159.)$$

Now, the area of the circular orbit is $= c$; therefore the above found area is to the area of the circle,

$$\text{as } \frac{1}{1 - 5n^{\frac{1}{2}}} + \frac{n}{1 - 5n^{\frac{3}{2}}} \text{ to } 1 :: 1 - 4n : 1 - 5n^{\frac{3}{2}}.$$

Hence the time of moving from apogee to apogee, is to the time of describing the circle ADE with the force r or $1 :: 1 - 4n : 1 - 5n^{\frac{3}{2}}$.

The

The periodic time of the body, in the curve ABC, with the compound force, will be nearly the same as in a circle, at the mean distance, and with the compound force proper to that distance. The least distance is r , or 1, the greatest distance =

$$1 + \frac{2n}{1 - 5n}, \text{ half the sum of these } = 1 + n,$$

nearly, equal the mean distance. The compound force, at this distance, will be easily found =

$$\frac{1 - n \times \overline{1 + n}^3}{1 + n^2}. \text{ It is a property of central forces}$$

commonly known, that the periodic times in circles, are in the subduplicate ratio of the radius of the circle directly, and of the central force inversly: Therefore, the time of describing the circle ADE, is to the time of describing a circle at the mean distance, or the time of a revolution, as

$$\sqrt{1 - n \times \overline{1 + n}^3} \text{ to } \overline{1 + n}^{\frac{3}{2}}. \text{ Hence, com-}$$

pounding this proportion with that found above, we shall have the time from apogee to apogee, to the time of a revolution round the body T, as $1 - 4n$

$$\times \sqrt{1 - n \times \overline{1 + n}^3} \text{ to } \overline{1 - 5n}^{\frac{3}{2}} \times \overline{1 + n}^{\frac{3}{2}},$$

or as $1 - \frac{2}{3}n + \frac{1}{3}n^2$ to $1 - 6n - \frac{1}{3}n^2$, nearly. Therefore if n be given, the motion of the apses may be found; and, if the motion of the apses be given, n is easily found: For, taking the same numbers as are made use of in prop. 9, sect. 1. Supplement to Tracts Physical and Mathematical, 58091: 57600 :: $1 - \frac{2}{3}n + \frac{1}{3}n^2$: $1 - 6n - \frac{1}{3}n^2$, or, $\frac{11199}{11199} = 1 - \frac{2}{3}n - \frac{1}{3}n^2$; hence n is easily found.

The same by a different method.

We found before that $u = \frac{y^{-\frac{1}{2}} \dot{y}}{\sqrt{2n-1-5n \times y}}$.

Then by similar triangles $TB : TG :: Bd : Gg$,

that is, $1+y : 1 :: \frac{y^{-\frac{1}{2}} \dot{y}}{\sqrt{2n-1-5n \times y}} : \frac{y^{-\frac{1}{2}}}{1+y}$

$\times \frac{\dot{y}}{\sqrt{2n-1-5n \times y}} = Gg$, which is the mea-

sure of the angle BTC. Throw $\frac{y^{-\frac{1}{2}}}{1+y}$ into a se-

ries, then, multiplying and finding the fluent, we

have $\frac{c}{\sqrt{1-5n}^{\frac{1}{2}}} - \frac{nc}{\sqrt{1-5n}^{\frac{3}{2}}} + \frac{3n^2c}{2.\sqrt{1-5n}^{\frac{5}{2}}} -$

$\frac{5n^3c}{2.\sqrt{1-5n}^{\frac{7}{2}}} \&c. =$ arc described in passing from

perigee to apogee; therefore, from what has been said above, it is evident, that $58091 : 57600 ::$

$\frac{c}{\sqrt{1-5n}^{\frac{1}{2}}} - \frac{nc}{\sqrt{1-5n}^{\frac{3}{2}}} + \frac{3n^2c}{2.\sqrt{1-5n}^{\frac{5}{2}}} \&c. : c,$

from which proportion, n will easily be found.

COROL-

COROLLARY.

It is a thing commonly known, and which we shall not stay here to demonstrate, that the disturbing force of the sun upon the moon is, *ceteris paribus*, in the direct simple ratio of the moon's distance from the earth; therefore n may very fitly be put equal the mean disturbing force found in Cor. 1st, Prop. 3d. Hence, for determining a , or the sun's

distance, there is this equation $\frac{p^2}{p^2} \times$

$$\frac{8a^4 + 20a^2 + 297}{16a^4} = n, \text{ which reduced gives } a =$$

$$\sqrt{\frac{15p^2}{16P^2n-8p^2}} + \sqrt{\frac{15p^2}{16P^2n-8p^2}}^2 + \frac{297p^2}{16P^2n-8p^2}.$$

But if Dr. Stewart's mean force be put $= n$, we shall have, when reduced as above, $a =$

$$\sqrt{\frac{45p^2}{16P^2n-8p^2}} + \sqrt{\frac{45p^2}{16P^2n-8p^2}}^2 + \frac{729p^2}{16P^2n-8p^2}.$$

These two equations will give the values of a much different, let n be what it will, as is evident from inspection.

SCHOLIUM.

We shall now endeavour to point out the principal mistakes, which, we apprehend, Dr. Stewart has made, in attempting to solve this curious problem.

And first, let us examine his calculations, supposing all the principles, he has gone upon, to be right.

In this proposition we had the following proportion, viz. the time of the body moving from apogee to apogee, is to the time of the body, describing the circle ADE, with the force of the body T alone

$:: 1 - 4n : \sqrt{1 - 5n}^{\frac{3}{2}}$. Then, if we suppose, with Dr. Stewart, that the time of a revolution of the body, in the curve ABC, is equal to the time of a revolution, in the circle AGD, with the compound force at A, we shall have the following proportion, which is the same with his; see page 381, *Traacts Physical and Mathematical*, viz. The time of describing the circle, with the force of the body T alone; is to the time with the compound force ::

$\sqrt{1 - n}^{\frac{1}{2}} : 1$. Therefore, compounding this proportion with the above; the time of moving from apogee to apogee, is to the time of revolving with

the compound force :: $1 - 4n \times \sqrt{1 - n}^{\frac{1}{2}} : \sqrt{1 - 5n}^{\frac{3}{2}}$;

that is, $58091 : 57600 :: 1 - 4n \times \sqrt{1 - n}^{\frac{1}{2}} :$

$\sqrt{1 - 5n}^{\frac{3}{2}}$; or (putting $a = 58091$, $b = 57600$)

$a^2 : b^2 :: 1 - 9n + 24n^2 - 16n^3 : \sqrt{1 - 5n}^3 = 1 - 15n + 75n^2 - 125n^3$. From dividing the

consequents by the antecedents, we get $\frac{b^2}{a^2} =$

$1 - 6n - 3n^2$, nearly; whence n will be found $= .0028015$.—Dr. Stewart, for finding the value of n , has this proportion, a^2 is to b^2 as the cube of the greatest distance is to the cube of the least distance.

distance. [See page 381, Tracts, &c.] But 1 is the least distance, and $1 + \frac{2n}{1-5n}$ the greatest dis-

tance; hence this proportion, $a^2 : b^2 :: 1 + \frac{2n}{1-5n}$

$: 1^2 :: 1 - 3n : 1 - 5n^3 :: 1 - 9n + 27n^2 - 27n^3 :$

$1 - 5n^3$. Compare this with the former proportion, and it will be easy to see where they differ. For, if n^3 in each be neglected, they only vary in the third term, where there is $27n^2$ in the last proportion, instead of $24n^2$ in the former. The difference is, indeed, very small, yet it occasions a very great alteration in the sun's distance, as will appear immediately. From this last proportion, the value of n is $\cdot 00279770$ &c. whose reciprocal is $357\cdot 43365$, the very same number found in prop. 9, sect. 1, supplement to Tracts, &c. Now if this value of n be substituted in the second equation for a , page 31, and p^2 be put $= 1$, and $P^2 = 178\cdot 725$, a will come out $= 495\cdot 932$ &c. which falls within the limits the Doctor has prescribed, and agrees with his number in every place, except the last. But if the other value of n , viz. $\cdot 0028015$, be substituted in the same equation, a will come out $= 89\cdot 8$, nearly.

The difference of the two values of n , here made use of, is only $\cdot 0000038$ &c. which is scarce the 737th part of the whole, and yet the difference in the sun's distance occasioned by this extremely small variation in n , or the mean disturbing force, is no less than 406 times the moon's distance; which is even more than the sun's distance has generally been thought to be. But if n were taken $= \cdot 00279759$,

and substituted in the same equation, a would come out infinite, though the variation here, from Dr. Stewart's value of n , is only .00000011 &c, which is less than the 25433d part of it. This last value

of n is $\frac{1}{357.45} = \frac{1}{2P^2}$, hence in the equation

for the value of a , the denominator $16P^2n - 8p^2 = 0$, therefore a must be infinite.

To prevent any objections, that may be made to the above method of calculation, we shall examine in what manner the value of n is affected, by rejecting all the terms in the value of v^2x^2 , in which y is found above the second power. [See page 381.]

Our value of u is $\frac{y}{\sqrt{2ny - 1 - 5n \times y^2}}$; but if

the term where y^3 is found be taken in, then u will

be $\frac{y}{\sqrt{2ny - 1 - 5n \times y^2 + 4ny^3}}$. Extracting

the root, the denominator becomes $=$

$$\frac{1}{2ny - 1 - 5n \times y^2}^{\frac{1}{2}} \times \frac{4ny^3}{2ny - 1 - 5n \times y^2}^{\frac{1}{2}} =$$

&c. Dividing y by this series, u will $=$

$$\frac{y}{\sqrt{2ny - 1 - 5n \times y^2}} - \frac{2ny^3y}{2ny - 1 - 5n \times y^2}^{\frac{3}{2}}$$

&c. Multiply both sides by $1 + y$, and take the fluent

fluent when $y = \frac{2n}{1 - 5n}$, and we have the whole

$$\text{area} = \frac{c}{\sqrt{1 - 5n}^{\frac{1}{2}}} + \frac{nc}{\sqrt{1 - 5n}^{\frac{3}{2}}} - \text{some small}$$

quantity, which put $= s$. This area therefore is to

$$\text{the area of the circle, as } \frac{c}{\sqrt{1 - 5n}^{\frac{1}{2}}} + \frac{nc}{\sqrt{1 - 5n}^{\frac{3}{2}}} - s$$

is to c , or as $1 - 4n - s \times \sqrt{1 - 5n}^{\frac{3}{2}}$ is to

$\sqrt{1 - 5n}^{\frac{3}{2}}$. And the time from apogee to apogee,

is to the time of a revolution, in the circle AGD,

with the force of the body T alone, as $1 - 4n - s$

$\times \sqrt{1 - 5n}^{\frac{3}{2}}$ is to $\sqrt{1 - 5n}^{\frac{3}{2}}$. By proceeding in the

same manner as above, it will easily appear, that n will come out greater than before, though s be ever so small; and if y^4 be taken into the account, n will still be greater.

From what has been said, it is evident, that the value of n , determined above, is not greater than it ought to be, and consequently, that Dr. Stewart's value is too little, allowing all the principles, he has gone upon, to be right.

And by the way, does it not appear probable, from the Doctor's being satisfied with a proportion for determining n , which, though very near the truth, is nevertheless inaccurate; that he has not been well apprized of the great alteration in the sun's

sun's distance, which is occasioned by a very small one in the in the disturbing force?—This is mentioned, is a circumstance which leads to the most favourable apology, that can be made, for the confidence with which our author gives us a measurement, of the solar system, so much different from that which has been hitherto held, by our most eminent astronomers, as not egregiously wide of the truth.—To proceed :

But if our value of n , determined above, be substituted in the equation for determining a , deduced from what we call the true mean disturbing force, we shall have a , or the distance of the sun, still much less than before ; and, indeed, much less than the sun's distance was ever thought to be ; for it is only ≈ 50.1 times the distance of the moon, nearly.

Here, then, is a very large error, arising from a mistake in the Doctor's *calculus* ; it now remains, that we examine the principles themselves, which he has assumed, and see, whether we have any thing to expect from this method of finding the sun's distance or not. Those objections only will be mentioned which seem to be of the greatest weight, nor shall we give any calculations, as it will be sufficient to point out where we apprehend he has erred.

In the first place, Dr. Stewart supposes the body to describe a revolution in the curve ABC, in the same time that another body would describe the circle AGD, with the compound force at A ; but that this is not so, has been proved, page 382.

2dly, Is it not essentially necessary in this enquiry, that the proportion of the solar force to that of the earth upon the moon, be exactly found ? And, in determining this, is not the periodic time of the moon, round the earth, taken into the account, and the earth supposed to be the centre, round which the moon moves ? This, therefore, must occasion a considerable error, the earth and moon, as it is well known,

known, moving round their common centre of gravity; and consequently, a period of the moon will be accomplished in less time, than if the earth were the centre about which it revolved. To correct this properly, it will be necessary to know exactly the proportion of the quantities of matter in the earth and moon; a particular, I am afraid, we shall never be able perfectly to come at.

3dly, We have anticipated another capital objection, where we observed, that though the mean disturbing force were ever so exactly found, yet the distance of the sun could not be determined from it to any *great* degree of accuracy. [See page 234.]

4thly, Since the centre of gravity of the earth and moon moves forward, in the ecliptic; several degrees every revolution of the moon round the earth, the motion of the apses occasioned by this disturbing force, will be considerably different from what it would be, supposing the centre of gravity at rest, as we have done in this proposition. This requires no demonstration, being evident as soon as mentioned.

5thly, The true figure of the moon's orbit being an ellipsis, it must, with the same disturbing force, have a different motion of the apses to that of our supposed circular one; and to answer the question right, the motion of the apses ought to be determined in general for any excentricity whatever.—To see how the above considerations would succeed in calculation, I made a rude trial, but the result varied greatly both from Dr. Stewart's numbers, and those commonly adopted by astronomers.

6thly, But if the motion of the moon's apogee cannot be accounted for, upon the common principles of central forces, (as is the opinion of Mr. Machine) will not this be a more disheartning circumstance than any pointed out above?

From

From what has been shewn, it plainly appears, that the method of determining the sun's distance, pitched upon by our author, is very ill fitted for the purpose; it being scarce possible, I think, to accomplish it this way (that is, by the motion of the apses) though the calculations were made with the greatest accuracy. For the principles are too complex to make it possible to take every thing into the account which belongs to it; and unless that be done, since the smallest neglect occasions a very great error, the result will ever be much wide of the truth.

Neither is the motion of the nodes much better fitted than the motion of the apses for the solution of this problem. In the first place, this motion is slower, and therefore requires a more accurate calculus. And besides, the force MQ (see fig. 278, pl. 18.) which mostly contributes to the motion of the nodes, affects the distance of the sun very much, by a small variation. For if the value of it be put into fluxions, and reduced as was shewn above, we shall have the variation of this force to the variation

of the sun's distance, as $-4y^2 + r^2$ to $a - 2y$. Hence it is evident, that as the variation of the sun's distance bears so very great a proportion to the variation of the disturbing force; it is requisite that this force be determined to the *last* degree of accuracy. But this will be very difficult, if not impossible to come at; the figure of the lunar orbit, its inclination to the ecliptic, &c. &c. being so variable.

From what has been said, 'tis presumed, it may be safely concluded, that the distance of the sun will never be satisfactorily ascertained by the THEORY of GRAVITY.

ARTICLE XLVII.

Improved solution to Question 109, by Mr. LOWRY.

THE solution of this problem depends on the following curious property of the circle, which I believe is not generally known, namely, If BKCL (Fig. 458, Pl. 25.) be a circle given in magnitude and position, and A be a given point without it, a point I may be found within the circle, such, that if any straight line BIC be drawn through it to intersect the circle at B and C, and AB, AC be drawn; the square of AB will always be to the square of AC as BI to IC.

To find this point, through O, the centre of the circle, draw AOL, and find the point E such that the rectangle AOE may be equal to the square of the semidiameter of the circle; bisect AE at D, and divide the diameter LK at I, so that $LI:IK = LD:DK$, so shall I be the point that was to be found.

Draw DB, DC and on KL demit the perpendiculars BG, CF; then, since $LI:IK :: LD:DK$ by construction, $BD:DC = BI:IC$, by *prop. VI. of Stewart's Propositiones Geometricae*; therefore ID bisects the angle BDC, and the right angled triangles BDG, CDF are similar; hence $GD:DF = BD:DC = BI:IC$. But by *Prop. VI. Stewart's General Theorems*, the square of AB, is equal to twice the rectangle AO·DG, and the square of AC equal to twice the rectangle AO·DF; therefore the square of AB is to the square of AC as DG to DF, that is, as BI to IC.

COROLLARY. The square of AL is to the square of AK as LI to IK.

Hence this CONSTRUCTION of the Problem proposed.

Through the centre O draw AKOL and divide AL at I so that LI may be to IK as the square of AL to the square of AK; on AI describe a semicircle and apply therein AQ equal to the given perpendicular. Draw BQIC, and join AB, AC, so shall ACB be the triangle required.

ARTICLE XLVIII.

To the Editor of the Mathematical Repository.

SIR,

THE advantage of transformation, previous to finding fluents, in many cases, is well known to those who are conversant in such speculations: for, by a judicious substitution, expressions of every complicated and perplexing appearance are brought to a more tractable form, whereby the labour of calculation is much facilitated. It is, therefore, a matter of importance, in this branch of the analytic art, to discover such a convenient substitution for some part of any fluxional expression, as shall tend to reduce it to the most simple form, which it is capable of assuming;—to point out a method for effecting this purpose, in certain cases, is the principal design of this essay. Besides, it has always appeared to me difficult for young fluxionists to comprehend the rationale of some of those fluents, which are commonly placed among the first in the tables, and, on that account, reckoned amongst the easiest: now, the want of perspicuity, in the commencement of any science, must have a great tendency to discourage young beginners, because obstacles present themselves, which cannot easily be surmounted at so early a period.

I have selected a few examples of such forms of fluxions as are most frequently met with, in the solution of problems, in order to illustrate a method of investigating their fluents, which, I believe, is not very generally known, and which, when it first occurred to my mind, I imagined had some degree of novelty in it. If you think it deserves a place in the Repository, by inserting it you will greatly oblige

Your most obedient Servant,

JAMES CUNLIFFE.

Bolton-le-Moors, Oct. 8, 1800.

EXAMPLE 1. To find the fluent of $\frac{x}{\sqrt{(x^2 - a^2)}}$?

Assume $x^2 - a^2 = (av - x)^2 = a^2v^2 - 2avx + x^2$, from

$$\text{whence } x = \frac{a}{2} \times \frac{v^2 + 1}{v}, \quad \sqrt{(x^2 - a^2)} = av - x = \frac{a}{2} \times \frac{v^2 - 1}{v},$$

$\frac{v^2-1}{v}$, and $\dot{x} = \frac{1}{2} av \times \frac{v^2-1}{v^2}$, and therefore $\frac{\dot{x}}{\sqrt{(x^2-a^2)}}$

$= \frac{\dot{v}}{v}$, which is the most simple of logarithmic forms, and its

fluent is well known to be the hyp. log. of v . But $x = \frac{a}{2} \times$

$\frac{v^2+1}{v}$, as reduced above, therefore, by solving a quadratic equa-

tion, $v = \frac{x + \sqrt{(x^2-a^2)}}{a}$, and consequently the fluent of

$$\frac{\dot{x}}{\sqrt{(x^2-a^2)}} = \text{h. l. of } \frac{x + \sqrt{(x^2-a^2)}}{a}.$$

EXAMPLE 2. To find the fluent of $\frac{\dot{x}}{\sqrt{(x^2-2ax)}}$?

Assume $x^2-2ax = (av-x)^2 = a^2v^2 - 2avx + x^2$, from

whence $x = \frac{a}{2} \times \frac{v^2}{v-1}$, $\sqrt{(x^2-2ax)} = av-x = \frac{a}{2} \times$

$\frac{v^2-2v}{v-1}$, and $\dot{x} = \frac{av}{2} \times \frac{v^2-2v}{(v-1)^2}$, and therefore $\frac{\dot{x}}{\sqrt{(x^2-2ax)}}$

$= \frac{\dot{v}}{v-1}$. Now the fluent of $\frac{\dot{v}}{v-1}$, is well known to be

the hyp. log. of $(v-1)$. But $x = \frac{a}{2} \times \frac{v^2}{v-1}$, as

determined above, whence $v = \frac{x + \sqrt{(x^2-2ax)}}{a}$, or $v-1$

$= \frac{x-a + \sqrt{(x^2-2ax)}}{a}$; and therefore the fluent of

$$\frac{\dot{x}}{\sqrt{(x^2 - 2ax)}} = \text{fluent of } \frac{\dot{v}}{v-1} = \text{h. l. } (v-1) = \text{h. l. } \left(\frac{x-a + \sqrt{(x^2 - 2ax)}}{a} \right).$$

The preceding fluent might have been easily obtained, by writing $x - a$ in the place of x in the first example; but I rather chose to investigate it at full length, as the conclusions will be found of use, in what follows.

The preceding example may be done as follows.

Assume $x^2 - 2ax = z^2x^2$, from whence $x = \frac{2a}{1-z^2}$,

$\sqrt{(x^2 - 2ax)} = zx = \frac{2az}{1-z^2}$, and $\dot{x} = \frac{4az\dot{z}}{(1-z^2)^2}$; there-

fore $\frac{\dot{x}}{\sqrt{(x^2 - 2ax)}} = \frac{2\dot{z}}{1-z^2} = \frac{\dot{z}}{1+z} + \frac{\dot{z}}{1-z}$; the fluents

of which are $\text{h. l. } (1+z) - \text{h. l. } (1-z) = \text{h. l. } \left(\frac{1+z}{1-z} \right)$

But $x = \frac{2a}{1-z^2}$, whence $z = \frac{\sqrt{(x^2 - 2ax)}}{x}$; conse-

quently the fluent of $\frac{\dot{x}}{\sqrt{(x^2 - 2ax)}} = \text{h. l. } \left(\frac{1+z}{1-z} \right) = \text{h. l. } \left(\frac{x + \sqrt{(x^2 - 2ax)}}{x - \sqrt{(x^2 - 2ax)}} \right).$

$$\left(\frac{x + \sqrt{(x^2 - 2ax)}}{x - \sqrt{(x^2 - 2ax)}} \right).$$

EXAMPLE 3. To find the fluent of $\frac{\dot{x}}{x\sqrt{(x^2 - a^2)}}$?

By making use of the values of x &c. as determined in the first example, there will be had $\frac{\dot{x}}{x\sqrt{(x^2 - a^2)}} = \frac{a}{2} \times \frac{\dot{v}}{1+v^2}$; the

fluent

fluent of. which is known to be $\frac{a}{2} \times$ circular arc to radius 1

and tangent $v = \frac{x + \sqrt{x^2 - a^2}}{a}$.

EXAMPLE 4. To find the fluent of $\frac{x}{x \sqrt{x^2 - 2ax}}$?

Taking the values of x &c. as determined in the first part of the second example, and there will be had $\frac{x}{x \sqrt{x^2 - 2ax}}$
 $= \frac{2v}{av^2}$; the fluent of which is $-\frac{2}{av} = -\frac{2a}{x + \sqrt{x^2 - 2ax}}$
 $= \frac{\sqrt{x^2 - 2ax} - x}{x}$, by restoring the value of v , as determined in the said example.

The same otherwise. Making use of the values of x &c. as determined in the second operation of the second example, there

will be had $\frac{x}{x \sqrt{x^2 - 2ax}} = \frac{z}{a}$; the fluent of which is $\frac{z}{a}$
 $= \frac{\sqrt{x^2 - 2ax}}{ax}$.

EXAMPLE 5. To find the fluent of $\frac{xx}{\sqrt{x^2 - 2ax}}$?

Taking the values of x &c. as determined in the former part of the second example, and there will be had $\frac{xx}{\sqrt{x^2 - 2ax}} =$

$\frac{av^2v}{2(v-1)^2} = \frac{az}{2} \times \frac{(z+1)^2}{z^2} = \frac{az}{2} + \frac{az}{z} + \frac{az}{2z^2}$; the fluents

of which are $\frac{az}{2} + a \times \text{h. l. } (z) - \frac{a}{2z}$, in which expression

restoring $z = v - 1 = \frac{x - a + \sqrt{(x^2 - 2ax)}}{a}$, and the

fluent of $\frac{xx}{\sqrt{(x^2 - 2ax)}}$ will be obtained.

EXAMPLE 6. Required the fluent of $\frac{2ax}{x\sqrt{(a^2 + x^2)}}$?

Assume $a^2 + x^2 = (av - x)^2 = a^2v^2 - 2avx + x^2$, from
whence $x = \frac{a}{2} \times \frac{v^2 - 1}{v}$, $\sqrt{(a^2 + x^2)} = av - x = \frac{a}{2} \times$
 $\frac{v^2 + 1}{v}$, and $\dot{x} = \frac{av}{2} \times \frac{v^2 + 1}{v^2}$; therefore $\frac{2ax}{x\sqrt{(a^2 + x^2)}} =$
 $\frac{4v}{v^2 - 1} = \frac{2v}{v - 1} - \frac{2v}{v + 1}$. Now the fluent of this last ex-
pression is known to be $2 \times \text{h. l. } (v - 1) - 2 \times \text{h. l. } (v + 1)$
 $= 2 \times \text{h. l. } \left(\frac{v - 1}{v + 1}\right)$. But $x = \frac{a}{2} \times \frac{v^2 - 1}{v}$, as deduced
above, from whence, by solving a quadratic equation, there will
be had $v = \frac{x + \sqrt{(a^2 + x^2)}}{a}$; and consequently the fluent of

$$\frac{2ax}{x\sqrt{(a^2 + x^2)}} = 2 \times \text{h. l. } \left(\frac{v - 1}{v + 1}\right) = 2 \times \text{h. l. } \left(\frac{x - a + \sqrt{(a^2 + x^2)}}{x + a + \sqrt{(a^2 + x^2)}}\right).$$

The preceding Example done something differently.

Assume $a^2 + x^2 = (xu - a)^2 = x^2u^2 - 2axu + a^2$, whence

$$x = \frac{2ax}{u^2 - 1}, \sqrt{(a^2 + x^2)} = xu - a = a \times \frac{u^2 + 1}{u^2 - 1}, \text{ and } \dot{x} =$$

=

$$= -2au \times \frac{u^2 + 1}{(u^2 - 1)^2}. \text{ Therefore } \frac{2ax}{x\sqrt{(a^2 + x^2)}} = -$$

$\frac{2u}{u^2 - 1}$, the fluent of which is $2 \times \text{h. l. } \left(\frac{1}{u}\right)$. But $x =$

$$\frac{2ax}{u^2 - 1}, \text{ as found above, whence } u = \frac{a + \sqrt{(a^2 + x^2)}}{x};$$

and therefore the fluent of $\frac{2ax}{x\sqrt{(a^2 + x^2)}} = 2 \times \text{h. l. } \frac{1}{u} = 2$

$$\times \text{h. l. } \left(\frac{x}{a + \sqrt{(a^2 + x^2)}}\right).$$

EXAMPLE 7. Required the fluent of $\frac{2ax}{x\sqrt{(a^2 - x^2)}}$?

Assume $a^2 - x^2 = (xu - a)^2 = x^2u^2 - 2axu + a^2$, whence

$$x = \frac{2ax}{u^2 + 1}, \sqrt{(a^2 - x^2)} = xu - a = a \times \frac{u^2 - 1}{u^2 + 1}, \text{ and } x$$

$$= -2au \times \frac{u^2 - 1}{(u^2 + 1)^2}; \text{ therefore } \frac{2ax}{x\sqrt{(a^2 - x^2)}} = -\frac{2u}{u^2 + 1},$$

the fluent of which is well known to be $2 \times \text{h. l. } \left(\frac{1}{u}\right)$. Now

$x = \frac{2ax}{u^2 + 1}$, as is found above, from whence, therefore, by a

quadratic equation, $u = \frac{a + \sqrt{(a^2 - x^2)}}{x}$, and consequently the

fluent of $\frac{2ax}{x\sqrt{(a^2 - x^2)}} = 2 \times \text{h. l. } \left(\frac{1}{u}\right) = 2 \times \text{h. l. } \left(\frac{x}{a + \sqrt{(a^2 - x^2)}}\right).$

$$\left(\frac{x}{a + \sqrt{(a^2 - x^2)}}\right).$$

EXAMPLE 8. To find the fluent of $\frac{2ax}{x\sqrt{(2ax-x^2)}}$?

Assume $2ax - x^2 = v^2x^2$, whence $x = \frac{2a}{1+v^2}$, $\sqrt{(2ax-x^2)}$

$= vx = \frac{2av}{1+v^2}$, and $\dot{x} = -\frac{4av\dot{v}}{(1+v^2)^2}$; therefore

$\frac{2ax}{x\sqrt{(2ax-x^2)}} = -2v$, the fluent of which is evidently —

$2v$. But $x = \frac{2a}{1+v^2}$, as deduced above, whence $v^2 = \frac{2a}{x}$

$- 1 = \frac{2ax - x^2}{x^2}$, or $v = \frac{\sqrt{(2ax-x^2)}}{x}$. Consequently the

the fluent of $\frac{2ax}{x\sqrt{(2ax-x^2)}} = -2v = \frac{2\sqrt{(2ax-x^2)}}{x}$.

EXAMPLE 9. To find the fluent of $\frac{x\dot{x}}{\sqrt{(2ax-x^2)}}$?

Taking the values of x &c. as determined in the preceding example, and there will be had $\frac{x\dot{x}}{\sqrt{(2ax-x^2)}} = -\frac{4v\dot{v}}{(1+v^2)^2} =$

$-2v \times \frac{1-v^2}{(1+v^2)^2} - 2v \times \frac{1+v^2}{(1+v^2)^2}$, the fluents of

which are $\frac{2v}{1+v^2} = 2A$; — where A denotes the length of a

circular arc to radius 1 and tangent $v = \frac{\sqrt{(2ax-x^2)}}{x}$; or, by

restoring the value of v , the expression becomes $\frac{\sqrt{(2ax-x^2)}}{a}$

$= 2A$, which is, therefore, the fluent of $\frac{x\dot{x}}{\sqrt{(2ax-x^2)}}$.

Diophantine Algebra has generally been esteemed a more curious than useful speculation; but from the foregoing examples it appears, that it is capable of being employed with success in the calculation of fluents, one of the most important branches of Mathematical learning.

ARTICLE XLIX.

ATWOOD'S INVESTIGATIONS ON WATCH BALANCES.

(Concluded from page 246.)

NOW let Q (fig. 82, pl. 5. and fig. 86, pl. 6.), or the point of quiescence of the auxiliary spring deviate from O , the point of quiescence of the balance spring, by an arc OQ ; suppose this arc OQ to be $= 1^\circ$; and let the point Q be situated in the first semiarc of vibration between O and B . The time of describing the semi-arc BO will be ascertained on these conditions, by referring to the investigation (page 129), and making the following substitutions :

$$a = 1.5707963 = \text{an arc of } 90^\circ \text{ to radius} = 1^*$$

$$b = 2.3561945 = \text{an arc of } 135^\circ$$

$$c = 2.3387412 = \text{and arc of } 134^\circ$$

$$b - c = d = 0.0174533 = \text{an arc of } 1^\circ$$

$$l = 193$$

$$f = 0.9562754$$

$$n = \frac{1}{2}.$$

shewed twelve minutes by the motion of the hands in one hour of mean time, which corresponds to an interval of 12 seconds of time, shewn by the watch in 60 seconds, or one minute of mean time. According to the calculation, 13 seconds of time are shewn by the watch in one minute. A nearer agreement between the theory and matter of fact could scarcely be expected in the circumstances of the experiment.

* In all the following calculations the radius is also $= 1$.

Then,

Then, $\sqrt{\frac{2a}{lf \times n+1}}$ \times into a circular arc,

of which the sine is $\sqrt{\frac{c \times n+1}{2b+2nc}}$ $\overset{\text{Pts. of a sec.}}{=} 0.0995507$

$\sqrt{\frac{a}{2lf}}$ \times into a circular arc, of which

the sine is $\frac{d}{\sqrt{b^2+nc^2}} = 0.00047174$

Time of a semivibration in the arc BO $= 1.0002244$

The time shewn by the watch in $24^h = \frac{24^0}{1.0002244}$

$= 23^h 59' 40''.60$, giving a daily rate of $19''.40$ flow.

This variation of the daily rate is not to be considered as effecting the regularity of the watch, as it is either compensated by adjustments when the watch is regulated to mean time, or taken as the established rate. A more material point is next to be determined; admitting the deviation of O and Q (fig. 82, and 86,) to be $OQ = 1^\circ$, the same as in the former case; suppose the femiarc of vibration to be diminished from 135° to 125° . If the points of quiescence O and Q were coincident, this diminution of the arc of vibration would cause no alteration (page 164, Vol. I.) in the time of a semivibration, because the forces of acceleration of both springs would be as the angular distances from the same quiescent point O; but since these points are separated by an arc of 1° , a diminution of 10° in the

the femiarc of vibration, will cause a change in the daily rate, which will be obtained from the general theorem (page 129) by making the following substitutions :

$$\begin{aligned} a &= 1.5707963 = \text{an arc of } 90^\circ \\ b &= 2.1816616 = \text{an arc of } 125^\circ \\ c &= 2.1642083 = \text{an arc of } 124^\circ \\ b - c &= d = 0.0174533 = \text{an arc of } 1^\circ. \\ l &= 193 \\ f &= 0.9562754 \\ n &= \frac{1}{28}. \end{aligned}$$

$$\sqrt{\frac{2a}{lf \times n + 1}} \times \text{into a circular arc,}$$

Pts. of a sec.

of which the sine is $\sqrt{\frac{c \times n + 1}{2b + 2nc}} = 0.09951472$

$$\sqrt{\frac{a}{2lf}} \times \text{a circular arc, of which the}$$

sine is $\frac{d}{\sqrt{b^2 + nc^2}} = \underline{\underline{0.00050949}}$

The time of a femivibration $\cdot 1.0002421$

Time shewn by the watch in $24^h =$

$$\frac{24^h}{1.0002421} = 23^h 59' 39'' .08$$

giving a daily rate of $20'' .92$ flow.

Daily rate, when the femiarc of vibration was $135^\circ 19'' .40$ flow.

Retardation of the daily rate in consequence of the diminution of the femiarc of vibration from 135° to 125° $\underline{\underline{1'' .52.}}$

In

In these examples, the point of quiescence of the auxiliary spring Q is situated in the first semiarc of vibration between B and O ; and in consequence of this position, the daily rate of the time-keeper is retarded by a separation of the points of quiescence, while the semiarc of vibration remains the same; i.e. 135° ; secondly, the separation of the points of quiescence remaining unaltered, a diminution of the arc of vibration causes a further retardation of the rate. Opposite effects on the daily rate take place when the point of quiescence Q is situated in the latter semiarc of vibration, (fig. 83.) between O and E . The arc OQ remaining $= 1^\circ$ as before, this case produces an acceleration instead of a retardation of the rate.

Suppose the semiarc of vibration BO to be 135°
 Separation of the points of quiescence $= OQ$ 1°

Arc BQ in fig. 83, or BN in fig. 86, 136°

The time of describing the semiarc BO will be determined by referring to the theorem (page 135),

making $a = 1.5707963 = \text{an arc of } 90^\circ$

$b = 2.3561945 = \text{an arc of } 135^\circ$

$c = 2.3736478 = \text{an arc of } 136^\circ$

$d = 0.0174533 = \text{an arc of } 1^\circ$

$l = 193.$

$f = 0.9562754.$

$n = \frac{1}{16}.$

The time of describing the semiarc $BO =$

$$\sqrt{\frac{2a}{lf \times n + 1}} \times \text{a circular arc of which the}$$

fine

Pts. of a sec.

fine is $\sqrt{\frac{b-d \times n + 1}{2b + 2nc}} = 0.09950617$

$\sqrt{\frac{a}{2lf \times 1 + 2n}}$ \times a circular arc of

which the fine is $\sqrt{\frac{d^2 \times 1 + 2n}{b^2 + nc^2 - 2nd^2}} = 0.00047141$

Time of a semivibration in the arc BO $= 0.09997758$

Time shewn by the watch in $24^h =$

$\frac{24^h}{0.9997758} = 24^h 0' 19'' .38,$

giving a daily rate of $19'' .38$ fast.

Every thing else remaining, let the semiarc of vibration be diminished from 135° to 125° , and make

$b = 2.1816616 = \text{an arc of } 125^\circ$

$c = 2.1991149 = \text{an arc of } 126^\circ$

The time of describing BO will now be

$\sqrt{\frac{2a}{lf \times n + 1}}$ \times a circular arc of which the fine is

Pts. of a sec.

$\sqrt{\frac{b-d \times n + 1}{2b + 2nc}} = 0.09946662$

$\sqrt{\frac{a}{2lf \times 1 + 2n}}$ \times a circular arc of

which the fine is $\sqrt{\frac{d^2 \times 1 + 2n}{b^2 + nc^2 - 2nd^2}} = 0.0050912$

Time of a semivibration in the arc BO $= 0.9997574$

Time

Time shewn by the watch in $24^h =$

$$\frac{24^h}{9997574} = \quad \quad \quad 24^h 0' 20'' \cdot 91,$$

giving a daily rate of $20'' \cdot 91$ fast.

Daily rate when the semiarc of vibration was 135° $19'' \cdot 38$ fast.

Acceleration of the rate in consequence of the diminution of the semiarc of vibration from 135° to $125^\circ = 1'' \cdot 53$

The theorem from which this latter result has been calculated is founded on a supposition, that the arc BN (fig. 83.) is described by the accelerative forces of the balance spring and the auxiliary spring u , and that the arc NO is described by the accelerative forces of the two springs just mentioned, combined with the retarding force of the other auxiliary spring v , of which the point of quiescence is situated at N (fig. 83.). It may possibly be thought, that this supposed action of the auxiliary spring which retards the balance while it describes the arc from N to O, cannot take place, because by the peculiar construction of Mr. Mudge's invention, when the balance vibrates from B towards E, the retarding force of the auxiliary spring is removed from acting on the balance while it describes the arc $Nk = 27^\circ$ (fig. 83.), including the said arc NO; but it is to be considered, that this removal of the retarding force from acting on the balance while it describes the arc from N to k , is merely a mechanical expedient for supplying the power which is lost by friction, and as it is thus wholly employed and counteracted, it appears requisite, in making calculations of the times in which the balance vibrates, that the entire forces of the springs are to be taken into the calculation in the same manner as if friction did not exist, and no mechanical contrivance was necessary for compensating

tating the loss of motion in consequence of it. But since, in point of fact, no retarding force acts on the balance while it describes the arc NO, according to the conditions under consideration, it may be satisfactory to calculate the time of a semivibration in the arc BO on this ground also, that is, on a supposition that the entire semiarc BO is described by the joint accelerative forces of the balance spring, and the auxiliary spring u only; it will appear that this slight variation of the conditions does not at all affect the conclusions deduced from the preceding calculations, and very little alters the results themselves.

The time of describing the semiarc BO, on these conditions, will be determined by referring to the theorem (page 130), and making the following notations.

$$a = 1.5707963 = \text{an arc of } 90^\circ \text{ to radius } = 1$$

$$b = 2.9561945 = \text{an arc of } 135^\circ$$

$$c = 2.3736478 = \text{an arc of } 136^\circ$$

$$d = 0.0174533 = \text{an arc of } 1^\circ.$$

$$l = 193$$

$$f = 0.9562754$$

$$n = \frac{1}{28}.$$

$$\text{The time of a semivibration is } = \sqrt{\frac{2a}{lf \times n + 1}}$$

\times into a circular arc, of which the sine =

Pts. of a sec.

$$\sqrt{\frac{b \times n + 1}{2b + 2nc}} = .9997752.$$

Time shewn by the watch in $24^h =$

$$\frac{24}{.9997752} = 24^h 0' 19'' .42,$$

giving a daily rate of $19'' .42$ fast.

VOL. II.

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Every

Every thing else remaining, let the femiarc of vibration be diminished from 135° to 125° ; and make

$$b = 2.1816616 = \text{an arc of } 125^\circ \text{ to radius 1.}$$

$$c = 2.1991149 = \text{an arc of } 126^\circ.$$

The time of describing the femiarc BO will now be =

$$\sqrt{\frac{2a}{lf \times n + 1}} \times \text{a circular arc, of which the}$$

Pts. of a sec.

$$\text{fine} = \sqrt{\frac{b \times n + 1}{2b + 2nc}} = 0.09997570$$

Time shewn by the watch in $24^h =$

$$\frac{24^h}{0.9997570} = 24^h 0' 20'' \cdot 96,$$

giving a daily rate of $20'' \cdot 96$ fast.

daily rate, when the femiarc of vibration was $135^\circ = 19'' \cdot 42$ fast.

Acceleration of daily rate in consequence of the diminution of the femiarc of vibration from

135° to $125^\circ = 1'' \cdot 54$,
scarcely differing from the former determination in page 406.

By having recourse to the figure, we may distinctly perceive the three quiescent positions of the arms GO and IO, (fig. 86.) in respect to the rods LM, ZW, which correspond with the several conditions assumed for the calculation of the preceding examples. In the first place, the crank AXYD, and the arms GO, IO, being in their quiescent position, it is evident from the preceding observations, that when the arms GO, IO are adjusted so as just to touch the rods LM, ZW, the points of quiescence of the auxiliary springs u and v coincide with the point of quiescence of the balance spring O, (fig. 84.)

84.) and in this case, the balance making 5 vibrations in a second when adjusted to mean time, the daily rate is $\equiv 0$. (page 136).

Secondly, suppose the arms GO and IO to be so affixed to the axis TR, FS, (fig. 86 and 82.) that instead of just touching the rods LM and NW when quiescent, they are inclined * to that position at any angle $OGQ = OIN$: the situation of the pallets p and q not being altered: in this case the point of quiescence of the auxiliary spring u , will be separated from the point of quiescence of the balance spring by the angular distance $OGQ = OCQ$. In like manner the point of quiescence of the auxiliary spring v will deviate from the point of quiescence of the balance spring by the angular distance $OIN = OCN$. Suppose that the arc $OQ = ON$ is $\equiv 1^\circ$: the point of quiescence Q of the auxiliary spring u is in the first semiarc of vibration, that is, between B and O, (fig. 82.) while the balance vibrates from B to E; and the point of quiescence of the auxiliary spring v , is in the first semiarc of vibration between E and O, while the balance vibrates from E to B; on these conditions, the semiarcs of vibration BO and OE being 135° , the daily rate of the time-keeper will be $19'' \cdot 40$ slower than mean time, (p. 402) and if the semiarcs BO, OE should be diminished from 135° to 125° , the daily rate will be $20'' \cdot 92$ slow (p. 403); the retardation of the daily rate in consequence of the diminished semiarc of vibration being $1'' \cdot 52$.

To consider the remaining case; if the arms GO, IO, (fig. 86.) are adjusted so as to press equally by the force of the auxiliary springs against the rods LM, WZ, when quiescent, the equal and contrary pressures prevent any apparent effect on the position

* The force of the main spring not being supposed to act on the balance wheel, or pallets p and q .

of the balance or crank $AXYD$; if the rod LM should be removed (by turning the balance in its plane), suppose that the arm GO rests in a position GN , at a distance beyond O , which is measured by the arc $ON = 1^\circ$; in this position the point of quiescence of the auxiliary spring u will be situated in the latter semiarc of vibration at N , between E and O , and by a similar adjustment of the quiescent position of the arm IO , (fig. 83 and 86.) the point of quiescence of the auxiliary spring v will be situated at Q in the latter semiarc of the vibration, between O and B ; in consequence of this position of the points of quiescence, the daily rate of the watch will be accelerated $19'' \cdot 38$, (p. 405) while the semiarc of vibration continues 135° , and when the semiarc of vibration is diminished to 125° , the daily rate will be further accelerated; the rate being $20'' \cdot 91$ fast, (page 406) or $1'' \cdot 53$ faster than when the semiarc of vibration was 135° .

From these several * results it may be concluded, that although the rate of going of Mr. Mudge's time-keeper depends materially on the quiescent position of the arms GO , IO , (fig. 86.), that is, on the position of the points of quiescence of the auxiliary

* It is scarcely necessary to observe, that although these results have been investigated from supposing the elastic forces of the spiral springs to be in the precise law of the tensions, or distances from quiescence; yet if the spring's forces should deviate somewhat from that law, the general conclusions deduced from the preceding calculations, respecting the acceleration or retardation of the rate arising from an alteration in the position of the points of quiescence, will still be true, although the degree in which these effects take place may not be exactly the same as when the spring's forces are in the precise law assumed in the investigations.

Supposing, as in the former examples, the points of quiescence, of the auxiliary springs to be at the distance of 1° from the point of quiescence of the balance spring, the variations of rate from mean time, when the semiarcs of vibration are 135° , 125° , 60° , and 10° severally, will be as expressed underneath:

Points

auxiliary springs, yet while that position remains unaltered, whatever it may be, the regularity of the time-keeper will not be affected in consequence of the said position, provided the semiarc of vibration continues the same; but when this arc is diminished, an acceleration or retardation of the daily rate will take place, according to the situation of the points of quiescence of the auxiliary springs referred to the semiarc of vibration in which these springs respectively act. If the points of quiescence of the auxiliary springs should be situated in the first semiarc of vibration, a diminution of these arcs will cause a retardation of the rate; but if the points of quiescence should be situated in the latter semiarc of the respective vibrations, the daily rate will be accelerated.

Points of quiescence in the first semiarc of vibration.

Semiarc of vibration.	Variation of the daily rate from mean time.			
135°	-	-	-	— 19'' 40
125°	-	-	-	— 20'' 92
60°	-	-	-	— 44'' 33
10°	-	-	-	— 4' 24'' 24

Points of quiescence in the latter semiarc of vibration.

135°	-	-	-	+ 19'' 38
125°	-	-	-	+ 20'' 91
60°	-	-	-	+ 43'' 60
10°	-	-	-	+ 4' 23'' 60

If the auxiliary springs should be applied for the sole purpose of making the vibrations of the balance isochronal, it may probably be found convenient to adjust the points of quiescence of the auxiliary and balance springs at a greater distance than 1°; suppose the distance to be 10°. Let the arc of vibration be 45°, or $1\frac{1}{2}$ of a revolution, and suppose the balance to make four vibrations in a second; on these conditions two auxiliary springs, each of which is about $\frac{1}{40}$ part of the force of the balance spring, or perhaps a single auxiliary spring proportionally stronger, will, in most cases, be sufficient to compensate for the want of isochronism in the balance spring.

By means of Mr. Mudge's construction, we may apply the principles deduced from the preceding investigations to correct such alterations in the daily rate as may arise from a diminished or increased arc of vibration. For if it should be known, from any satisfactory mode of trial, that the properties * of the balance spring are such as cause the longer arcs of

* Different opinions have been entertained respecting the times in which a balance, vibrating freely by the action of a spiral spring, describes the longer and shorter arcs of vibration. Mr. Harrison says, "large arcs are naturally performed in less time than small ones."—Notes taken at the discovery of Mr. Harrison's time-keeper, p. vii. In this opinion he is followed by M. Berthoud; "J'ai appris par des expériences sûres, que les grand arcs et les petits arcs d'un balancier ne sont pas isochrones, et qu'en general dans un balancier libre, les grands arcs sont plus prompts que les petits." Mr. Ludlam, in his report addressed to the commissioners of the board of longitude, and intitled, a Short View of the Improvements made or attempted in Mr. Harrison's Watch, has the following remark: "The principle on which Mr. Harrison forms the alteration of the third part (before described) is, that the longer vibrations of a balance moved by the same spring are performed in less time. This is contrary to the received opinion among philosophers and workmen."

These contradictory opinions may possibly have arisen from experiments, in which all these circumstances capable of influencing the times of vibration in longer or shorter arcs, were either not noticed, or omitted to be properly allowed for; this will seem the more probable if it be allowed, that a balance spring may be adjusted in various ways so that either the longer or shorter arcs shall be performed in the least time; not only by altering the thickness and strength of the spring in different parts, but, if we subscribe to the opinion of M. BERTHOUD, in a spring uniform in every respect throughout, by altering the length and number of turns. He infers, that a certain length and number of turns may be given to an uniform spiral spring which will make it perfectly isochronal. This latter principle, however, does not appear to have been verified by any satisfactory experiments. According to the inferences deduced from the preceding investigations, three spiral springs, which are not isochronal when acting singly, may be so united by properly adjusting their points of quiescence, that their combined action shall cause the balance to perform its vibrations in any two arcs of unequal lengths in the same time.

vibration

vibration to be described in less time than the shorter arcs ; whenever the arc of vibration is diminished, the time-keeper will lose ; and this error in the rate would be corrected by adjusting the points of quiescence of the auxiliary springs in the respective latter semi-arcs of vibration at such a distance $ON = OQ$ (fig. 83 and 86.) from the point of quiescence of the balance spring, as corresponds with the error intended to be rectified.* In like manner, if the property of the balance spring should be such that a diminution of the arc of vibration causes an acceleration of the daily rate, this error will be corrected by placing the points of quiescence of the auxiliary springs in the first semi-arcs of vibration, (fig. 82 and 86.) at their proper distances from the point of quiescence of the balance spring. It is, however, to be remarked, that if the balance spring should be of the latter kind here assumed, and the points of quiescence of the auxiliary springs, either by any casual derangement in their position, or by an adjustment of them for the purpose of experimental observation, should be placed in the latter semi-arcs of vibration, the effect of this position would be an acceleration of the rate, whenever the semi-arc of vibration is diminished : and this effect would be produced on a double account : first, from the assumed nature of the balance spring, which disposes it to describe the smaller arcs in less time than the larger arcs of vibration : and secondly, from the position of the points of quiescence of the auxiliary springs. But it is evident from the preceding considerations, that although the balance spring should not be isochronal, yet the regularity of the

* The position of the points of quiescence of the auxiliary springs is here understood to be altered, by affixing the arms GO, IK, differently on the axes TR, ES ; the quiescent position of the pallets being no ways changed.

time-keeper will not be at all affected, however the points of quiescence of the auxiliary springs may be situated in respect of the point of quiescence of the balance spring, as long as the semiarc of vibration continues unchanged; and if the semiarc of vibration should be liable to increase or diminution, Mr. Mudge's construction affords an effectual remedy against this cause of variation in the rate, since the arms projecting from the auxiliary springs may be adjusted so, as to place their points of quiescence either in the first or latter semiarc of vibration, according as the balance spring, when acting singly, causes the shorter or longer vibrations to be described in the least time.

ARTICLE L.

Answers to the Mathematical Questions proposed in ARTICLE XXXIV. No. IX.

I. QUESTION 183, answered by Mr. David Davis, Warminster.

PUT $r = 1.00013368$, the amount of 1l. and its interest for one day; $a = 5000l.$; $n = 2555$, the number of days in 7 years; and $x =$ the daily expenditure. Then the amount of a for one day is ra , therefore $ra - x$ will represent what remains at the end of the first day, the amount of which is $r^2a - rx$; therefore $r^2a - rx - x$ is the sum remaining at the end of the second day, the amount of which is $r^3a - r^2x - rx$. Hence

it is evident that $r^n a - r^{n-1}x - r^{n-2}x - r^{n-3}x \dots \dots$

$- r^2x - rx - x$ will represent the sum remaining at the end of the n th, or 2555th day, which by the question must be equal to

nothing; that is, $r^n a - x \times (r^{n-1} + r^{n-2} + r^{n-3} \dots \dots$

$+ r^2 + r + 1) = 0$. But $r^{n-1} + r^{n-2} + r^{n-3} \dots \dots + r^2 +$

$a + r + 1$ is a geometrical progression, the sum of which is easily found to be $(r^n - 1) \div (r - 1)$; therefore $r^n a = x \times (r^n - 1) \div (r - 1) = 0$. Hence $x = ar^n \times (r - 1) \div (r^n - 1) = 2.3119971. = 21. 6s. 2d. 3.5 q.$ the sum to be expended daily.

If $n = 7$, instead of 2555, then r being $= 1.05$, the yearly expenditure will come out 854.0981. which being divided by 365 gives 21. 7s. 4d. nearly for the daily expenditure in this case.

The same, answered by Mr. G. Buffham, Boston.

Let r , a , n and x represent as before. Then $\frac{x}{r}$ will be the present worth of x pounds to be expended the first day, $\frac{x}{r^2}$ the present worth of x pounds to be expended the second day, and so on. Consequently the present worth of all the daily expenditures will be defined by $\frac{x}{r} + \frac{x}{r^2} + \frac{x}{r^3} + \dots + \frac{x}{r^n}$, which, by the question, must be equal to a ; that is, $x \times (\frac{1}{r} + \frac{1}{r^2} + \frac{1}{r^3} + \dots + \frac{1}{r^n}) = a$. Whence $x = a \times (r - 1) \div (1 - \frac{1}{r^n}) = a \times (r - 1) \times r^n \div (r^n - 1)$, the same as before.

The same, answered by Mr. John Blackwell, Hungerford.

Neat solutions were also
Buffham, Johnson, $t = 7$, $r = 1.05$, and $x =$ the annuity.
Squire.

V Then, by Simpson's Algebra, pa. 237, $p \times r^t =$ the amount of p in the time t ; also $x \times \frac{r^t - 1}{r - 1} =$ the amount of the

the annuity x , in the same time t . Hence, by the question, $x \times$

$\frac{r^t - 1}{r - 1} = p \times r^t$, and $x = 864.099731. = 864l. 1s. 11d\frac{1}{2}$, the

365th part of which is 2l. 7s. 4d. nearly the daily expenditure required.

According to one or other of these methods was the answer given by Messrs. Collins, Lowry, Marrat, May, Merones Minor, and Thornoby.

II. QUESTION 184, answered by Mr. John Johnson, Birmingham.

Put $a = 88$ yards the versed sine of the segment, $b = 9680$, the square yards in two acres, and $x =$ the diam. of the circle. Then, by Dr. Hutton's Mensuration, Rule 6th, for finding the area of a circular segment, $\frac{1}{4}a\sqrt{(xa - \frac{1}{2}a^2)} = b$, this reduced gives $x = 130.14375$, therefore 131.14375 is the diam. of a circle whose circumference is the outward boundary of the circular part of the wall, the versed sine of which is $88 + 1 = 89$; hence the area is found, by the above rule, $= 9868.989$, from which deducting 9680 , the area of the garden, we have 188.989 square yards, area of the base of the wall, $= 1700.901$ square feet, which multiplied by 7, the height, gives the content of the wall $= 11948.307$ solid feet.

Other answers were also received from Messrs. Barron, Blackwell, Buffham, Davis, Lowry, Marrat, May, Merones Minor, Squire, and Thornoby.

III. QUESTION 185, answered by Miss Susan May.

It is evident, from Art. 3388 of Martin's Institutions, that the force of gravity, at the equator, is to that, at the pole, as 230 to 231, and that the increase, in any latitude, is nearly as the square of the sine of that latitude; therefore, if the force, at the equator, be 1, the increase of gravity, at the pole, will be $\frac{1}{230}$, and (radius)² : (sine)² 30°, or $1 : \frac{1}{4} :: \frac{1}{230} : \frac{1}{880}$, Hence the force, in latitude 30°, is to $1 - \frac{1}{880} = \frac{879}{880}$ as 921 to 920.

Similar to this is the proposer's solution.—By Emerson the end of graphy, Sec. I. Prop. X. Cor. 2nd, the force of gravity at ϕ equal to is $\frac{2}{3}\frac{1}{10}$, when that at the equator is 1, and the diminution there at the equator $= \frac{1}{3}\frac{1}{10}$. Hence, by Cor. 1st, to the same proposition, $1^2 : (\sqrt{3} \div 2 = \cos. \text{lat.})^2 :: \frac{1}{3}\frac{1}{10} : \frac{1}{880}$, the diminution in the lat. 30°, which subtracted from $\frac{2}{3}\frac{1}{10}$ leaves $\frac{221}{880}$ for the force required.

The

The same, answered by Mr. John Barron, Schoolmaster, Spilby.

If g denotes the gravity at the equator ($= 1$), and y the gravity under any other latitude λ (30°), then $y = (1 + 0.0052848 \sin^2 \lambda) \times g$. See Dr. Hutton's Dictionary, Art. Gravity.

Ingenious solutions were received from Messrs. Blackwell, Buffham, Johnson, Lowry, Marrat, Squire, and Thornoby.

IV. QUESTION 186, *answered by Mr. Thomas Hewit, Teacher of the Mathematics, Featherstone-street, London.*

Let r = the radius of the earth, and R = that of the moon's orbit. Put $p = 3.14159$ &c. and $d = 16 \frac{1}{11}$ the space, through which a heavy body, at the surface of the earth, descends in the first second of time, the $2d$ will be the measure of the force of gravity at the surface, therefore the velocity, per second, in a circle at the surface of the earth will be $\sqrt{2rd}$, and the time of a revolution $= p \times 2r \div \sqrt{2rd} = p\sqrt{(2r \div d)}$ seconds. Because the force of gravitation, above the surface, is known to vary according

to the square of the distance inversely, we have $r^{-\frac{1}{2}} : R^{-\frac{1}{2}} ::$

$\sqrt{2rd}$, the velocity per second at the surface, to $(r^{\frac{1}{2}} \div R^{\frac{1}{2}})$

$\sqrt{2rd} = r\sqrt{2d \div R}$ the velocity in a circle whose radius is $R = 5183.3$ feet, nearly, per second. Also, because the squares of the periodic times vary as the cubes of the distances directly, the periodic times will vary as the square roots of the cubes of the

distances, therefore $r^{\frac{3}{2}} : R^{\frac{3}{2}} :: p\sqrt{(2r \div d)}$, the periodic time,

at the surface, to $(R^{\frac{3}{2}} \div r^{\frac{3}{2}}) \sqrt{(2r \div d)} = p \times (R \div r) \times \sqrt{(2R \div d)}$, the time of a revolution in a circle, whose radius is $R = 640041.9$ seconds. See Simpson's Fluxions, Vol. I. pa. 240, 241.

Neat solutions were also received from Messrs. Barron, Blackwell, Buffham, Johnson, Hill, Marret, May, Merones Minor, and Squire.

V. QUESTION 187, *answered by Merones Minor.*

Fig. 459, Pl. 25. Let NBG be the base of the cone, CE the length of the shadow, C being the centre; draw EB a tangent to the circle at B, and CB, CA \perp to EB, EC respectively. Then NA is a quadrant $= 3.75$, and $CA = r = 7.5 \div 3.14159$;
put

put $x = \text{arch AB}$, then $AB + BE = \text{arch} + \text{its cotangent} = 18 - 3.75 = 14.25 = b$. Now, by Emerson's Trig. B. I. Prop.

$$15, \text{ Cor. 2nd, the cotan. of AB} = \frac{r^2}{a} - \frac{x}{3} - \frac{x^3}{45r^2} - \frac{2x^5}{945r^4}$$

$$\&c; \text{ therefore } \frac{r^2}{a} - \frac{x}{3} - \frac{x^3}{45r^2} - \frac{2x^5}{945r^4} \&c. = b; \text{ and, by}$$

$$\text{reversion of series, } x = \frac{r^2}{b} + \frac{2r^4}{3b^3} + \frac{13r^6}{15b^5} + \frac{1818r^8}{945b^7} \&c. =$$

$$.40772; \text{ Then } BE = b - x = 13.84228, \text{ and } EC = \sqrt{(BE^2 + BC^2)} = 14.04664.$$

Again, by Mayer's Tables, the sun's declination, at the given time, was $14^\circ 11' 22.5''$ N, hence, per spherics, the sun's altitude was $29^\circ 36' 2.1''$, and then, radius: tang. sun's alt. $\therefore EC : 7.9798$,

$$\text{the height of the cone; and finally, } \frac{15^2}{4\rho} \times \frac{\text{height}}{3} \text{ (putting } \rho =$$

$$3.14159 \&c.) = 47.626 \text{ feet, nearly, the content required.}$$

The same, answered by Mr. Thomas Squire, Baldock.

Let C be the centre of the cone's base, ABD a quadrant of its circumference, ABE part of the string $= 14.25$ feet, B the point where the string quits the circumference, and E the extremity of the cone's shadow.—Make $C\alpha = C\beta = 1$, and draw the tangent line $\beta\epsilon$ parallel to BE. Then, as 15 feet, the circumference of the cone's base, is to 3.14159×2 , so is ABE $= 14.25$, to $\alpha\beta\epsilon = 5.960305 =$ the sum of the arc $\alpha\beta$ and its cotang. $\beta\epsilon$ to radius 1. Now, I think the best though indirect method to find the $\angle ACB$ is, to assume some \angle near the true one, then by the 10th and 12th of Dr. Hutton's Mathematical Tables, after a few trials, the $\angle ACB (= CEB)$ will be found $= 9^\circ 47' 7\frac{2}{3}''$, hence, as sine of $9^\circ 47' 7\frac{2}{3}'' : BC = 2.3873262 ::$ radius 1 : $CE = 14.04655$ the length of the shadow from the centre of the cone's base. Then, by having the latitude day and hour given, the sun's altitude is found $= 29^\circ 36' 1\frac{1}{2}''$ or by allowing for refraction, semidiameter, and parallax, the true altitude of the sun's upper limb $= 29^\circ 53' 25\frac{1}{2}''$. Then, as cosine of $29^\circ 53' 25\frac{1}{2}'' : CE ::$ sine of $29^\circ 53' 25\frac{1}{2}'' : 8.0769$ the altitude of the cone. Hence the content $= 48.20558$ feet.

The

The same, answered by Mr. John Johnson, Birmingham.

Let ABHGN represent the circular base of the cone, and EBANGHE the chord; let AG be a diam. drawn \parallel to HB, the line joining the points of contact, and let C be the centre of the circle. Join CB and CE and make $BC \perp$ to AC. Then the circumf. GNA is $= 7\frac{1}{2}$ feet, which taken from 36 feet, the length of the cord, leaves $28\frac{1}{2}$ feet, the half of which is $14\frac{1}{2}$ = the sum of the arc AB and tangent BE. Put $x = BE$, and $r =$ rad. of the circle $= 2.39$, then $\sqrt{(x^2 + r^2)} = CE$, and by sim. Δ s, $CE : BE :: CB : BF = xr \div \sqrt{(x^2 + r^2)} = Cc$, and $Ac = r - xr \div \sqrt{(x^2 + r^2)}$. Hence, by rule 5th of Dr. Hutton's Mensuration, 8vo. Ed. the length of the arc AB is $= 2r \times$

$$\sqrt{\frac{3\sqrt{(x^2 + r^2)} - 3x}{5\sqrt{(x^2 + r^2)} + x}}. \text{ Therefore } 2r \times \sqrt{\frac{3\sqrt{(x^2 + r^2)} - 3x}{5\sqrt{(x^2 + r^2)} + x}}$$

$+ x = 14\frac{1}{2}$. By the method of trial and error x is found $= 13.81$, and $CE = \sqrt{(BE^2 + BC^2)} = 14.05$. Now, from the data, I find the sun's altitude $= 29^\circ 36'$, and thence the content of the cone 47.5 feet nearly.

The same, answered by Mr. Wm. Francis.

Let NE represent a right line drawn from the ends of the chord, through the centre of the cone's base C, and terminating with its edge at N. Then will BE be a tangent to the cone's base, which suppose $= 10$. Now the circumference being 15, the radius of the base will be 2.38736 ; whence $CE = 10.281$. And as $CE : \text{rad.} :: BE : \text{fine } \angle BGE = 76^\circ 34' 20''$ Therefore $\angle BCN = 103^\circ 25' 40''$ consequently the length of the arch BAN $= 4.30949$. But $4.30949 + 10 = 14.30949$, which should be 18, therefore the error is 3.69051 , too little. Again, if we suppose $BE = 15$, the error will be 1.126736 , too much; whence, by the rule of Position, $BE = 13.83$ very nearly, and $CE = 14.034$. Now, the sun's alt. corrected for the given time, was $29^\circ 56'$; hence, as cosine $29^\circ 56' : 14.034 :: \text{fine } 29^\circ 56' : 8.08$ feet; therefore the solidity was 48.223 feet.

According to one or other of these methods the question was answered by Messrs. Barron, Blackwell the proposer, Cunliffe, Hill, Lowry, Marrat, May, and Thornoby.

VI. QUESTION 188, answered by Mr. James Cunliffe, Bolton.

Let the fractions be denoted by x , y , and z . Then, by the question, $1 - xy = a^2$, and $1 - xz = b^2$, whence $x = (1 - a^2) \div y = (1 - b^2) \div z$, or $z = y \times (1 - b^2) \div (1 - a^2)$; by means of which, $1 - yz = 1 - y^2 \times (1 - b^2) \div (1 - a^2) = a$ square, per question. Assume $1 - ry$ for its root, that is, make $1 - y^2 \times (1 - b^2) \div (1 - a^2) = (1 - ry)^2 = 1 - 2ry + r^2y^2$, which being reduced gives $y = 2r \times (1 - a^2) \div (r^2 \times (1 - a^2) + (1 - b^2))$, where a , b , and r may be taken at pleasure.

Or the solution may be as follows. Put $1 - xy = a^2$, $1 - xz = b^2$, and $1 - yz = c^2$; by proceeding as before, $1 - y^2 \times (1 - b^2) \div (1 - a^2) = c^2$, which gives $y^2 = (1 - a^2) \times (1 - c^2) \div (1 - b^2)$. Whence it is plain that a , b , c may be expounded by any proper rational fractions of such a nature as that $(1 - a^2) \times (1 - c^2) \div (1 - b^2)$ may be a rational square, or that $1 - a^2$, $1 - b^2$, $1 - c^2$, may all be rational squares. It therefore appears, that three fractions may be found of such sort, that if the product of every two be subtracted from unity, the remainders shall be equal, respectively, to given squares. But the given squares must be subject to some such restrictions as those above specified.

Example. Put $a = \frac{2}{3}$, $b = \frac{4}{5}$ and $c = \frac{5}{13}$: then $y^2 = \frac{(1 - a^2)(1 - c^2)}{(1 - b^2)} = \frac{56}{169}$, or $y = \frac{16}{13}$; whence $x = \frac{13}{25}$, and $z = \frac{13}{15}$, being three fractions that will answer the question.

The same, answered by Miss Susan May.

Let x , y , and z represent the fractions required; then, by the question, $1 - xy$, $1 - xz$, and $1 - yz$ must be squares. Put $1 - xz = m^2$, and $1 - yz = n^2$. From the first of these equations $x = (1 - m^2) \div z$, and from the second $y = (1 - n^2) \div z$; therefore $1 - xy = 1 - (1 - m^2) \times (1 - n^2) \div z^2$, which must be a square, that is, $z^2 - (1 - m^2) \times (1 - n^2)$ must be a square. Assume its root $= z - r$, then $z^2 - (1 - m^2) \times (1 - n^2) = z^2 - 2rz + r^2$, and, by reduction, $z = \frac{(1 - m^2) \times (1 - n^2) + r^2}{2r}$, where m , n , and r must be fractions

less than unity.

If $m = \frac{1}{2}$, $n = \frac{1}{3}$, and $r = \frac{1}{6}$, then $z = \frac{31}{36}$, $x = \frac{27}{36}$, and $y = \frac{33}{36}$; hence $1 - xy = (15 \div 33)^2$, $1 - xz = (3 \div 6)^2$ and $1 - yz = (2 \div 6)^2$.

The

The same, answered by Mr. W. Marrat, Boston.

Let x , y , and z , denote the required fractions, and put $1 - xy = r^2$, $1 - xz = v^2$, and $1 - yz = w^2$; then, by transposition,

$$\left. \begin{array}{l} 1 - r^2 = xy \\ 1 - v^2 = xz \\ 1 - w^2 = yz \end{array} \right\} \text{and by resolving into factors } \left\{ \begin{array}{l} (1+r)(1-r) = x \times y \\ (1+v)(1-v) = x \times z \\ (1+w)(1-w) = y \times z \end{array} \right. \text{ according to Lellies' method}$$

$$\text{that is, by assumption } \left\{ \begin{array}{l} 1 - r = my \text{ and } 1 + r = x \div m \\ 1 - v = nz \text{ and } 1 + v = x \div n \\ 1 - w = pz \text{ and } 1 + w = y \div p \end{array} \right\} \text{ Hence}$$

$$\left. \begin{array}{l} 1 - my = r = x \div m - 1 \\ 1 - nz = v = x \div n - 1 \\ 1 - pz = w = y \div p - 1 \end{array} \right\} \text{ and } \left\{ \begin{array}{l} x = 2m - m^2 y \\ x = 2n - n^2 z \\ y = 2p - p^2 z \end{array} \right\} \text{ From these equations, by proper reduction, we get}$$

$$\left. \begin{array}{l} y = (2n^2 p + 2mp^2 - 2nm^2 p^2) \div (n^2 + m^2 p^2) \\ z = (2m^2 p - 2m + 2n) \div (n^2 + m^2 p^2) \\ x = (2nm^2 p^2 - 2n^2 m^2 p + 2mn^2) \div (n^2 + m^2 p^2) \end{array} \right\} \text{ If } m = 1, n = 3, p = 4 \text{ then } y = 8 \div 25, z = 12 \div 25, \text{ and } x = \frac{42}{25}$$

The same, answered by Mr. I. H. Swale.

Assume $v \div x$, $v \div y$, and $v \div z$ for the required fractions. Then the conditions of the question are, that $1 - v^2 \div xy$, $1 - v^2 \div xz$, and $1 - v^2 \div yz$ shall be perfect squares. That is, $(xy - v^2) \div xy$, $(xz - v^2) \div xz$, and $(yz - v^2) \div yz$ must be perfect squares, in order to which it is necessary that xy , xz , and yz must be squares. Hence, then, we have to determine three squares xy , xz , and yz such that subtracting the given square, v^2 , from each, the remainders shall be perfect squares. Let $xy - v^2 = m^2$, then $xy = m^2 + v^2$, a square also; let it be equal $r^2 + m^2 + 2rm$, then will $m = (v^2 - r^2) \div 2r$, $m^2 = (v^2 - r^2)^2 \div 4r^2$, and $xy = m^2 + v^2 = (v^2 - r^2)^2 \div 4r^2 + v^2 = (v^2 + r^2)^2 \div 4r^2$. In like manner we have $xz = (v^2 + s^2)^2 \div 4s^2$, and $yz = (v^2 + t^2)^2 \div 4t^2$, all squares, where v , r , s , and t may be taken at pleasure.

But, it yet remains to determine the respective values of x , y , and z . For brevity, put $(v^2 + r^2)^2 \div 4r^2 = a^2$, $(v^2 + s^2)^2 \div 4s^2 = b^2$, and $(v^2 + t^2)^2 \div 4t^2 = c^2$. Then $x = a^2 \div y = b^2 \div z$; that is, $y = z \times (a^2 \div b^2) = c^2 \div z$; or $x = b^2 c^2 \div a^2$, and $z = bc \div a$; hence $x = b^2 \div z = ab \div c$, and $y = c^2 \div z = ac \div b$. That is, by restoring the values of a , b , and c , $x = t \times (v^2 + r^2) \times (v^2 + s^2) \div 2rs \times (v^2 + t^2)$, $y = s \times (v^2 + r^2) \times (v^2 + t^2) \div 2rt \times (v^2 + s^2)$, and $z = r \times (v^2 + s^2) \times (v^2 + t^2) \div 2st \times (v^2 + r^2)$. Therefore, by proper substitution, the fractions become

$$\frac{2rst \times (v^2 + r^2)}{t \times (v^2 + r^2) \times (v^2 + s^2)}, \quad \frac{2rtv \times (v^2 + s^2)}{s \times (v^2 + r^2) \times (v^2 + t^2)}, \quad \text{and}$$

$$\frac{2stv \times (v^2 + r^2)}{r \times (v^2 + s^2) \times (v^2 + t^2)}.$$

Let $v = 1$, $r = 2$, $s = 3$, and $t = 4$ and the fractions will be $51 \div 50$; $32 \div 51$; and $6 \div 17$.

Messrs. Barron, Hill, Lowry, M'Doneld, and Merones Minor, sent neat solutions to this question.

VII. QUESTION 189, answered by Mr. Wm. Peacock, Birmingham.

Analysis. Let CF (fig. 460, Pl. 25.) represent the meridian, AC, CB the courses, and A, B the position of the ships, at the time when the rock at F is in a direct line between them. In the $\triangle ACB$ there are given, $\angle ACF = BCF = 16^\circ 52\frac{1}{2}'$, $AC - CB = 3.75$, and $CF = 9$ leagues.—About the $\triangle ACB$ let a circle be described and continue CF till it meets it at Q; draw the diam. QSD, which will be \perp to the base AB, and join DC, DB, and DF; and upon AC demit the \perp DE. Then, it is well known that, CE is $=$ to half the diff. of the sides, and DC is \perp to QC, theref. the $\triangle DCE$ is given in magnitude, consequently D is a given point, but F is a given point and the $\angle FBD = ACD$ is given, being the complement of half the given \angle : Hence this

CONSTRUCTION. On the line bisecting the given \angle take CF of the given length, and $EC =$ half the diff. of the sides; draw DC, DE \perp to CF, CA respectively, intersecting in D, and upon the line joining the points D, F, describe a segment of a circle to contain an $\angle =$ the complement of half the given one, and let it meet CB in B; draw BF and it is done.

CALCULATION. Let O be the centre of the circle and draw the radii OD, OF, and OB and on DF demit the \perp OP and produce it to meet CB in P' and CF in Y. Then in the right $\angle d \triangle CED$, EC and $\angle ECD$ are given to find DC, and in the right $\angle d \triangle DFC$, FC and DC are given to find DF and the $\angle DFC$. Again, in the right $\angle d \triangle s$ FPO, FPY, PF and the $\angle s$ FOP, PFY are known to find the sides FO, PO, FY, PY and the \angle FYP, hence YO becomes known, and also CY. In the $\triangle YCP'$, CY and all the $\angle s$ are known to find CP' and YP', then in the $\triangle BOP'$, OP' and OB are known to find BP' and the $\angle BOP'$, hence AC, CB are known, and in the isosceles $\triangle BOF$, $\angle BOF$, and BO, FO are known to find BF, and be-
cause

cause $CB : CA :: BF : AF$, AB is found $= 6.70956$, and the rates of sailing, per hour, $= 4.656$ and 3.156 leagues respectively.

The same, answered by Mr. James Cunliffe.

Fig. 461, Pl. 25. Let P represent the port, R the rock, A the place of the swifter vessel, B that of the other. It is evident, from the question, that in the $\triangle APB$, there are given $PR = 9$ leagues, being the length of the line bisecting the vertical $\angle APB$, $AP - BP = 3\frac{1}{2}$ leagues, and $\angle APR = BPR = 16^\circ 52\frac{1}{2}'$. Draw Rm parallel to the sides, also $mr \perp$ to PR . The $\triangle PmR$ is isosceles, wherefore, as radius : secant of $\angle mPr :: Pr = \frac{1}{2} PR : Pm = Rm = 4.7024874$, which put $= a$, also put $AP - BP = Am - Bm = 3.75 = d$, and $Bm = x$, then $Am = d + x$. The $\triangle s BmR, AmR$ are similar, therefore $Bm : Rm :: Rm : Am$, whence $Bm \times Am = (Rm)^2$, or $x^2 + dx = a^2$, which gives $x = \sqrt{(a^2 + \frac{1}{4}d^2)} - \frac{1}{2}d = 8.1875105$; whence $PB = 7.8899979$, and $PA = 11.6399979$. And it is easily found that the swifter vessel must have sailed at the rate of 4.656 leagues per hour, and the other at the rate of 3.15 leagues in the same time. AB or the distance of the two vessels at the time mentioned may be found by trigonometry and is 6.70927 leagues.

Or the solution might have been as follows :

Draw $RE, Re \perp$ to the sides PB, PA , and upon PA take $Pb = PB$, also take $eb' = eb$, and draw Rb', Rb . The $\triangle bRb'$ is isosceles, by the construction, therof. $\angle Ab'R = PbR = PBR$, and conseq. $\angle ARb' = \angle APB$, the given vertical \angle . In the right $\angle d \triangle PRb$, the hypotenuse PR is given, together with the $\angle RPe$, therof. Re becomes known; wherof. in the $\triangle Ab'R$, there are given the $\perp Re$, and $Ab' = PA - PB = PA - Pb$, the given difference of the sides, and $\angle ARb' = Ab'R - bAR = \angle APB$, which is the 15th problem in the Appendix to Simpson's Algebra, and the calculation may be performed, as there directed, without the aid of Algebra.

From the latter method of solving the preceding problem, it appears that, $\sqrt{}$, $S - s$, and L is reducible to $A - a$, B and P , or $\sqrt{}$, P and $m - n$; the former of which takes place when the $\angle B$ exceeds a right angle; and the latter data take place when both the angles A and B are acute.

The same, answered by Mr. John Barron, Spilby.

Fig. 461, Pl. 25. Since both ships had been out $2\frac{1}{2}$ hours, and the rate of sailing of the one exceeded that of the other by

$1\frac{1}{2}$ leagues per hour, if $x = PB$, the distance sailed by the least, the distance sailed by the other, that is, AP will be $= x + 3.75$, and the distance between the port and the rock, that is, PR is $= 9$ leagues; also half the vertical \angle , or half the \angle contained between the courses of the ship, is given $= 16^\circ 52' 30''$, the cosine of which call c . Then, by Prop. 6, Cor. 5, pa. 105 of Emerson's Trigonometry, as $2AP \times PB : PR \times (AP + PB) :: \text{radius} : \text{cosine of half } \angle APB$, that is, as $2x^2 + 7.5x : 18x + 38.75 :: 1 : c$, or, by multiplying means and extremes $2cx^2 + 7.5cx = 18x + 38.75$; hence $x = 7.889985 = PB$ and AP is $= 11.639985$, consequently the rates of sailing are 3.15599 , and 4.65599 leagues, per hour, and their distance, at the end of $2\frac{1}{2}$ hours, is easily found $= 6.70988$ leagues.

The same, answered by Mr. John Johnson.

Having, like Mr. Peacock, found that D is a given point, Mr. Johnson then draws the lines S, E , which is well known to be \parallel to CF , and therefore given by position, also the $\angle FSD$ being a right angle, the following *Construction* is evident.

Take $EC = \frac{1}{2}$ diff. of the sides, and draw $ED, CD \perp$ to AC, FC respectively, to meet at D ; draw $ES \parallel$ to CF , and on FD describe a semicircle meeting ES at E and draw FS to meet AC, CB , at A and B , and it is done; for AC, CB are the distances sailed by the two ships, and AB is their distance asunder.

Ingenious solutions were also received from Messrs. Blackwell, Buffham, Davis, Hill, Lowry, Marrat, May, Merones Minor, Squire, and Swale.

VIII. QUESTION 190, answered by Mr. Geo. Buffham, Boston.

Let AD (Fig. 462, Pl. 25.) be a portion of the horizon, AB a part of the plane making the $\angle BAD = 33\frac{1}{2}^\circ$. Draw $BC \perp$ to AB , and $AC \perp$ to AD , meeting in C , which will be the centre of gravity of the pyramid, as is evident from the principles of mechanics. It is also evident that $BC = \frac{1}{4}$ of the height of the pyramid $= 2\frac{1}{4}$, and $\angle ACB = BAD$, therof. by trigono. radius : $BC :: \tan. ACB : AB = 1.489243$. In fig. 463, pl. 25, let EFG represent the base of the pyramid, b its centre, a the point where the perpendicular from the centre of gravity meets the side, agreeing with the point A in fig. 462, and de a right line in the plane parallel to the intersection of that plane and the plane of the horizon. Then $ab = AB$, and $bc =$ the radius of the inscribed

inscribed circle: whence by trig. $ab : bc :: \text{rad.} : \text{cof. } \angle abc = 14^\circ 15' \frac{1}{2} = \angle eaG$ (since by the nature of the question ba is \perp to de) = the angle the side EG made with ed .

The same, answered by Mr. Wm. Marrat, Boston.

Fig. 464, Pl. 25. Let EPF represent the pyramid (whose axis $OP = 9$ feet, and each side of its base $EL, LF, FE = 5$ feet,) standing on the plane $ABCD$, which makes an angle (BAY) of $33^\circ \frac{1}{2}$ with the horizon; also let G be its centre of gravity, which is known to be at $\frac{1}{4}$ of its height, and let GA be \perp to the horizon, which, in the position of the solid, will pass through the edge of its base as at S . Join OS, OL, OE and through S draw QR parallel to the horizon, or at right angles to OS .

Then, as radius : $OG (2.25) :: \tan. \angle OGS (33^\circ \frac{1}{2}) : OS$; as $\text{fine } \angle EOL (120^\circ) : EL (5) :: \text{fine } \angle OEL (30^\circ) : OL$; and as $OS : \text{fine } \angle OLS (30^\circ) :: OL : \text{fine } \angle OSL = 75^\circ \frac{1}{2}$; hence $OSQ - OSL = 90^\circ - 75^\circ \frac{1}{2} = 14^\circ \frac{1}{2} = \angle LSQ$, the angle which one side of the pyramid's base made with a horizontal line QR .

The same, answered by Mr. Thomas Squire, Baldock.

Let ACB (Fig. 465, Pl. 25.) be the base of the pyramid standing on the inclined plane, D its vertex, and DO the axis drawn perpend. to the base, the point O is equally distant from the points A, B, C , and, by the property of the equilateral Δ , $AO = 2.8866$. Let G be the centre of gravity of the frustrum and draw $GF \perp$ to the horizon to meet the inclined plane at S and draw OSE . Now it is evident that the pyramid will support itself on the inclined plane while the line GF falls within its base, and therefore when it can just stand, the point S will fall in the side of the base AC . By a known property $OG = \frac{1}{4}OD = 2.25$ and $\angle OGS = \angle FEO = 33^\circ \frac{1}{2}$, therefore SO is found = 1.4892 , and in the ΔAOS , AO, SO and the $\angle SAO (= 30^\circ)$ are known; hence the $\angle ASO = 75^\circ 45'$, and its complement $14^\circ 15'$ is the angle of inclination required.

True solutions to this question were received from Messrs. Barron, Blackwell, Burdon, Cunliffe, Johnson, Hill, Lowry, May, M'Donald, Merones Minor, and Thornoby.

IX. QUESTION 191, answered by Miss Susan May.

Fig. 466. Pl. 25: Let BE be the upright piece and AB the horizontal one, and let AD be the oblique piece required.
Com.

Complete the parallelogram ABDF and on AD drop the \perp FG. Then if AD represent the absolute strength of the oblique piece, AF will be that part of it which is exerted against AB in the direction AF. Now, by Prop. 55, Emerson's Mechanics, 8vo. the strength of AD is reciprocally as its length; therefore, by the property of the lever, $AB \times AF \div AD$ must be a maximum: but $AB \times AF = DF \times AF = AD \times GF$, theref. GF must be a max. which it evidently will be when $AF = FD = AB = 4\frac{1}{2}$; therefore $AD = AB \sqrt{2} = 6.364$ feet the length required.

Also, by Mr. Wm. Franis.

Let BE represent the upright piece, and AB the horizontal. Then, since by the principles of Mechanics the forces on each are directly as the sines of their opposite angles, it follows, that these angles must be equal for AD to sustain the greatest weight possible. But equal angles are subtended by equal sides. Hence, make BD = to AB, then will AD be the piece whose length is required. And $\sqrt{AB^2 + AB^2} = 6.36396 = AD$.

The same, answered by Mesones Minor.

Suppose AB the horizontal piece, BE the upright piece, and AD the required oblique piece. Draw BG, GT \perp to AD, AB. Then, by Em. Mec. 8vo, Prop. 45, Cor. 2, the force acting at A in the direction DA, to turn AB about the point B, is as BG, and ibid. Prop. 55, the strength of AD is as $1 \div AD$; that is, the force required will be as $AB \times BG \div AD$, AB being given.

By sim. Δ s, $AD : AB :: BG : GT$, a max. by the question :

Hence this CONSTRUCTION. On AB describe a semicircle and from the centre T demit the \perp TG to meet the semicircle in G, then through G draw AGD, the position of the oblique piece required. Then $AD = AB \sqrt{2} = 6.364$ feet, nearly.

This question was also ingeniously answered by Messrs. Blackwell, Buffham, Cunliffe, Johnson, Hill, Lowry, Marrat, and Thornoby.

X. QUESTION 192, answered by James Cunliffe.

Fig. 467, Pl. 25. Let CSB be the semicycloid, CF its axis, ESD a tangent to the curve at the point S, meeting CF, FB in E and D respectively. Draw the semiordinate GS perpendicular to the axis, cutting the periphery of the circle in I, also draw SW

SW perpend. to FD. Put $CF = 2a$, $CG = x$, and arch $CI = z$. Draw the chord CI , which will be parallel to the tangent ES , per prop. 3. Emerson on the Cycloid, and per corol. to prop. 2. *ibid.* $GS = GI + \text{arch } CI = \sqrt{(2ax - x^2)} + z$. Then, by reason of the parallels CI and ES , $GI : GC :: GS : GE = (GC \times GS) \div GI = x(\sqrt{(2ax - x^2)} + z) \div \sqrt{(2ax - x^2)}$; also $GC : GI :: SW : WD = (GI \times SW) \div GC = (2a - x) \sqrt{(2ax - x^2)} \div x$. Now it appears from the *Scholium to Theorem 19, Simpson on the maxima et minima*, that the length FW of the greatest cylinder that can be inscribed in the solid formed by the revolution of the semicycloidal space CSB about the base FB will be $= \frac{1}{2}WD$, that is, $\sqrt{(2ax - x^2)} + z =$

$$\frac{2a - x}{2x} \sqrt{(2ax - x^2)}, \text{ whence } z = \frac{2a - 3x}{2x} \sqrt{(2ax - x^2)};$$

from which equation, by the method of trial and error, z , in this case will be found to be an arc of $31^\circ.5055 = 31^\circ 30\frac{1}{2}'$ nearly. By means whereof $GS = FW = 1.681609a$, the length of the cylinder, and $GF = WS = 1.622438a$, the semidiameter of its base, and therefore its solid content is $10.764725a^3$ nearly.

Moreover the length GF of the greatest cylinder that can be inscribed in the solid formed by the revolution of the semicycloidal space CSB about its axis CF will be $= \frac{1}{2}GE$: that is,

$$2a - x = \frac{x(\sqrt{(2ax - x^2)} + z)}{2\sqrt{(2ax - x^2)}}, \text{ whence } z = \frac{4a - 3x}{x} \sqrt{(2ax - x^2)},$$

from which equation, by the method of trial and error, z in this case will be found to be an arc of $83^\circ.9287 = 83^\circ 55' 722$, by means of which the length FG of the inscribed cylinder will be found to be $1.105766a$, and $GS = 2.4592032a$, the semidiameter of its base, and from these dimensions its solidity is found to be $21.0088824a^3$, which is almost double the content of the other. Therefore the greatest cylinder that can be cut out of the solid formed by the revolution of the semicycloid about its axis, exceeds the greatest cylinder that can be cut out of the solid formed by the revolution of the same semicycloid about its base by $10.2441574a^3$, where a denotes the radius of the cycloid's generating circle. If the proposer means that the diameter of the generating is 16, then $a = 8$, and $10.2441574a^3 = 5245.0085888$ the required difference of the cylinders.

§ 3. If any one should be inclined to examine the approximations in the preceding solution, the following observations may tend to facilitate the labour, and be of use in other cases of the kind when they occur in the solution of problems.

In the first case we have shewn that $x = \frac{2a - 3x}{2x} \sqrt{(2ax - x^2)}$,
 or arc CI $= \frac{2a - 3x}{2x} \times$ sine GI, when x denotes the versed sine,
 and a the radius of the arc CI. Now it is very obvious that any
 arc is greater than its sine, wherefore $\frac{2a - 3x}{2x}$ must be greater
 than unity; put it equal to $1 + v$, viz. $\frac{2a - 3x}{2x} = 1 + v$, this
 being reduced gives $x = \frac{2a}{5 + 3v} = \frac{4a}{10 + 6v}$; from whence it
 appears that because v is positive, x must be less than $\frac{4a}{10}$. And,
 in the same manner, in the second case, it may be shewn, that x
 must be less than a .

The same, answered by Miss Susan May.

Fig. 467. Pl. 25. Let ACB represent a section of the solid
 formed by the revolution of the cycloid about its axis CF, and
 TSWN a section of the greatest inscribed cylinder. At S draw
 the tangent SE, to meet the axis produced at E; then, by B. II.
 Prop. 5, Cor. Emerson on curve lines, when the cylinder is a max.
 the subtangent EG is $= 2GF$, and by B. I. Prop. 3, Cor. 2, ibid.
 $EG = (CG \cdot GS) \div GI$: but, by the nature of the cycloid IS
 is $=$ the arc IC; therf EG is $(CG \div GI) \times (GI + \text{arc IC}) =$
 $CG + (CG \div GI) \times \text{arc IC}$; conseq. $CG + (CG \div GI) \times$
 $\text{arc IC} = 2GF = 2FC - 2CG$; therf. $2CG + (CG \div GI)$
 $\times \text{arc IC} = 2FC =$ twice the diam. of the given circle.—Now
 to find the value of CG it is obvious that the arc IC divided by
 GI must be between 1 and 2: suppose it $= 1\frac{1}{2}$, then $4\frac{1}{2} \times CG$
 $= 2FC$, or $CG = \frac{2}{3}FC$ nearly. But to obtain a more accurate
 value of CG we must make use of some approximating theorem
 such as that on page 126, Dr. Hutton's Mensuration, where it is
 shewn that the length of the arc IC is $= FC \sqrt{[3CG \div (3CF -$
 $CG)]}$, and GI is $\sqrt{(CF - CG) \times CG}$, therf. $3CG +$
 $\frac{CG}{\sqrt{(CF - CG) \times CG}} \times FC \sqrt{\frac{3CG}{3CF - CG}} = 2CF$, this re-
 duced and put into numbers gives $3CG + \frac{16\sqrt{3} \times CG}{\sqrt{(768 - 64CG + CG^2)}}$
 $= 32$.

Again, let QLR (fig. 468, pl. 25.) be a section of the solid
 formed by the revolution of the cycloid about its base OL and
 NHKM a section of the cylinder. Then it appears as before that
 XP is $= 2OX$, and conseq. by sim. Δs ON is $= 2NY$, and
 $4NY = 2QN + (2QN \div NZ) \times \text{arc QZ}$; hence $QQ -$
 QN

QN is $= 2QN + (2QN \div NZ) \times \text{arc } QZ$, or $OQ = 3QN + (2QN \div NZ) \times \text{arc } QZ$. And by assuming the arc QZ divided by NZ $= 1\frac{1}{2}$ as before, we have, by using Dr. Hutton's Theorem, and proceeding as before,

$$3QN + \frac{3^2 \sqrt{3} \times NQ}{\sqrt{(768 - 64NQ + NQ^2)}} = 16.$$

From these two equations, by means of trial and error, Miss May finds the dimensions of the cylinders very near the same as Mr. Cunliffe.

XI. QUESTION 193, answered by Mr. I. T. M'Doneld.

First. To determine the centre of gravity of a conic frustum ABCD (fig. 469, pl. 25.). Put $AB = 2b$, $CD = 2d$, $d - b = h$, $SQ = a$, $SV = x$, $MN = 2b + 2hx \div a$, and $3 \cdot 1416 = c$. Then the fluxion of the solid $ABMN \times SV$ the distance of the centre of gravity of the generating plane from the axis of suspension, is $= b^2 c x x + (2b c h x^2 x) \div a + (c h^2 x^3 x) \div a^2$, whose fluent, divided by the solid, (when $x = a$), is $a \times (6b^2 + 8bh + 3h^2) \div (12b^2 + 12bh + 4h^2)$, which may serve as a general theorem for ascertaining the centre of gravity of a conic frustum. In applying it to the present question, we have $b = 1\frac{1}{2}$, $d = 2$, $h = \frac{1}{2}$, and $a = 20$, whence we find $SG = 10 \frac{3}{7}$ as required.

Secondly. To determine the centre of oscillation in the frustum of a right cone. Let P be the point of suspension in the axis produced, then retaining the above symbols, and putting $PS = e$,

$VR = y$, and $RT = [(b + hx \div a)^2 - y^2]^{\frac{1}{2}}$, we shall have

$[(e + x)^2 + y^2] \times 4y \times [(b + hx \div a)^2 - y^2]^{\frac{1}{2}}$ for the fluxion of the generating plane into $(PR)^2$, the square of its distance from the axis of motion; the whole fluent of which (x being constant) is $c \times (e + x)^2 \times (b + hx \div a)^2 + \frac{1}{2} c \times (b + hx \div a)^2$, and this expression being drawn into $SV (x)$, the fluent thereof; (when $x = a$), is

$(e^2 b^2 + b e^2 h + \frac{1}{2} e^2 h^2 + \frac{1}{2} d^2 + \frac{1}{2} d^2 h + \frac{1}{2} d^2 h^2 + \frac{1}{2} d h^2 + \frac{1}{2} h^2) \times ca + (e^2 e + \frac{1}{2} b e h + e h^2) \times ca^2 + (\frac{1}{2} d^2 + \frac{1}{2} d h + \frac{1}{2} h^2) \times ca^3$, which call F .

Again the fluxion of the solid into PV is

$$b^2 c e x + \frac{2b c e h x x}{a} + b^2 c x x + \frac{c c h^2 x^2 x}{a^2} + \frac{2b c h x^2 x}{a} \times \frac{c h^2 x^2 x}{a^2}$$

whose fluent, (when $x = a$), is $(b^2 e + 2b e h + c h^2) + ca + (\frac{1}{2} b^2 + \frac{3}{2} b h + \frac{1}{2} h^2) \times ca^2$, which put $= M$; also put the frustum $ABMN$

ABMN = B, and the distance of the centre of gravity from S, found as above, = g . Then, vide Emerson's Fluxions, pa. 311,

$$\frac{F}{M} = \frac{F}{(e+g)B} = \frac{+(e^2b^2+be^2h+\frac{1}{2}e^2h^2+\frac{1}{4}d^4+\frac{1}{2}d^2h^2+\frac{1}{4}dh^2+\frac{1}{20}h^4) \times ca}{(b^2e+2beh+eh^2) \times ca + (\frac{1}{2}d^2+\frac{1}{2}dh+\frac{1}{2}h^2) \times ca^2}$$

which is a general theorem for ascertaining the distance PO of the centre of oscillation in the frustum of a right cone from the point of suspension. In the case before us, when the frustum is suspended by the centre of the less end, e being = 0, all the terms in which it is found will vanish, and by substituting the values of the other letters as above, we get PO = 14 very nearly. Hence $\sqrt{14} : \sqrt{39.2} :: 60 : 100$, the number of vibrations made by the given frustum in a minute.

XII. QUESTION 194, answered by Mr. Robert Wallace, the Proposer.

Fig. 470. Pl. 25. Let the velocity of the point n be to that of the point m in the given ratio of a to b . Take AC a 4th proportional to a , b , and BC. Join any two contemporary positions of the points n and m . Then, because the motions are uniform, the spaces passed over will be as the velocities, that is, $Bn : Cm :: a : b$, but $a : b :: BC : CA$, by constr. Therefore $CB : CA :: Bn : Cm$, or $CB : Bn :: CA : Cm$. Consequently $CB - Bn : Bn :: CA - Cm : Cm$, or $Bn : nC :: Cm : mA$. Therefore the curve required is a parabola, touching CB, CA in the points B and A. Q. E. F.

Cor. 1. Let CA, CB be given in magnitude and position, and let it be required to describe a parabola that shall touch CB, CA in the points B and A.

Take $CB : CA :: Bn : Cm :: Bn' : Cm'$. Join nm , $m'n'$; the circles which circumscribe the triangles nCm , $n'Cm'$, will intersect each other in F, the focus of the parabola; the curve may therefore be described by the common methods.

Cor. 2. Let there be found in CB, CA, any number of points n , n' ; m , m' ; &c. such that $Bn : Cm$ (and $Bn' : Cm'$) :: CB : CA. The parabola which touches CB, CA in the points B, A, will also touch each of the straight lines so found by joining the points nm , $n'm'$, &c. and will divide them at the points of section and contact in a similar manner, that is,

$$Bn : nC :: nD : Dm, \text{ \&c.}$$

The same, answered by Mr. I. T. M'Donald.

Let the given lines CB, CA be of such lengths that while n describes BC, m describes CA, then will they be tangents to the curve at B and A; and let n and m be contemporaneous positions of those points, so will the right line nm , which joins them, be likewise a tangent to the curve, from the conditions of the problem. Now it is evident that $BC : CA :: Bn : Cm$, which is a property of the parabola. For, let F be its focus, and draw AF, BF, CF, mF , and nF ; then (*vide solution to question 88, vol. 2, page 54, Repository,*) the triangles BCF, ACF, are similar, as are the triangles BnF , CmF . Hence

$$BC : CA :: BF : CF :: Bn : Cm.$$

Neat solutions were also received from Messrs. Cunliffe, and Lowry.

XIII. QUESTION 195, answered by Mr. Tho. Hewitt, London.

ANALYSIS. Fig. 471, Pl. 25. Suppose ABC the Δ required. Produce BC to meet a semicircle described on the base AB in D, and join AD. Produce, also, the sides till CL, CM are each = to half the given perimeter. Now LO, MO, being made \perp to CL, CM, will evidently meet CE, the line bisecting the vertical \angle , produced, in the same point O. Demit the \perp s CG, OF, and FH. Then, by Eu. II. 12, $AB^2 = AC^2 + BC^2 + 2BCD$, but, by hyp. $AB^2 = AC^2 + BC^2 + ACB$; theret. rect. ACB = 2 rect. BCD, or $AC = 2CD$; hence the \angle ACD is given, being $\frac{1}{2}$ of two right angles, and conseq. its supp. the vertical \angle ACB is given; theret. the equal radii OL, OM, OF are given, and, by hyp. the ratio of $CE : EG = OE : EF = OF : FH$ is given: but OF is given, theret. FH is given, and so is, OH, and AB is \perp OF: Whence the construction is manifest.

COR. When the vertical \angle of a Δ is 120° , the sum of the squares of the sides together with their rectangle, is equal to the square of the base.

The same, answered by Mr. James Whalley, Bolton.

CONSTRUCTION. Fig. 471, Pl. 25. Make the \angle LCM = 120° , and CL, CM each = half the given perim. and make LO, MO \perp to CL, CM; from O describe the arch LM and join CO. Take $CE' : CO$ in the given ratio, and with the centre C and radius CE' describe the arch ab , and draw OE' a tangent thereto,

thereto, draw also AB parallel to CE' to touch the arch LM , and ACB is the Δ required.

For draw $CG \perp$ to AB , then by constr. we have $CO : CE' :: CE : EG$, also, by drawing $AD \perp$ to BC produced, the $\angle ACD = \angle CAD$, therefore $CA = \angle CD$. By Emerson's Geom. II. 22, $AC^2 + BC^2 + BC \times \angle CD = AB^2$, theref. $AC^2 + BC^2 + BC \times AC = AB^2$; the rest is evident from the construction.

The same, answered by Mr. Louis Hill.

CONSTRUCTION. Fig. 472, Pl. 25. Take $EF =$ the given perimeter and on it constitute the equilateral ΔEGF , and on GH , drawn \parallel to EF , take $GH : EG$ in the given ratio of the segment to the bisecting line. Draw $HC \perp$ to GH , and from G , to HC , apply $GC = GE$; join EC , FC , and draw CA , CB , making the $\angle ACE = \angle AEC$, and the $\angle BCF = \angle BFC$, so will ACB be the Δ required.

Let GC meet AB in D , and on AC produced drop the $\perp BS$, then because $GC = GE$, by constr. the $\angle s$ GEC , GCE are equal, and taking from these the equal $\angle s$ AEC , ACE , there remains the $\angle GEF = \angle GCA$. And in the same way it is shewn that the $\angle GFE = \angle GCB$; but the $\angle GEF = \angle GFE$, by constr.; theref. the $\angle GCA = \angle GCB$, and theref. GC bisects the vertical $\angle ACB$; wherefore, by parallels, $CD : DP :: CG (EG) : GH$, that is, in the given ratio. Again, because of the $\angle s$, AC is $= AE$ and $BC = BF$, theref. $AC + CB + AB = EF =$ the given perimeter. Moreover, since it is shewn above that the $\angle ACE$ is $=$ to twice the $\angle GEE$, that is $=$ to two thirds of two right angles, the $\angle BCS$ is $=$ to one third of two right angles, that is, the ΔBCS is one half of an equilateral one, theref. $CB = \angle CS$; and by Euc. II. 12, $AB^2 = AC^2 + CB^2 + AC \cdot \angle CS$, but $\angle CS \cdot AC = CB \cdot AC$, therefore $AB^2 = AC^2 + CB^2 + AC \cdot CB$.

Very neat solutions were also received from Messrs. Cunliffe, Lowry, McDoneld, Peacock, and Swale.

XIV. QUESTION 196, answered by Mr. James Whalley, Bolton.

ANALYSIS Fig. 473, Pl. 25. Let ACB represent the Δ , about which describe a circle, and draw the diam. $EF \perp$ to AB . Let fall the $\perp CD$, which produce till it cuts the circle again in G , and join AG , BG .

By

By the quest. the ratio of $CD^2 : ADB = CD^2 : CDH = CD : DH$ is given. Moreover, $\text{rect. } ACB = EF \cdot CD$ and $AGB = EF \cdot DG$, by Simpson's Geometry, III. 25. Wherefore, the ratio of $EF \cdot CD : EF \cdot DG$, that is, the ratio of $CD : DG = ACB : AGB$, but the ratio of $CD : DG$ has been shewn to be given, therefore the ratio of the $\text{rect. } ACB : AGB$ is given, and the $\text{rect. } ACB$ is given, therof. the $\text{rect. } AGB$ is given. Again $MG^2 - BG^2 = AD^2 - DB^2 = AC^2 - CB^2$, by Simp. Geom. II. 9; whence, the rectangle and the diff. of the squares of AG and BG being given, AG and BG will each be given. Finally $CD^2 - DG^2 = CB^2 - BG^2 = \text{a given magnitude}$, because CB and BG , are both given, whence as the ratio and the diff. of the squares of the two lines CD, DG are both given, these lines will each be given, by means of which the problem may be readily and easily constructed.

The same, answered by Mr. John Lowry.

ANALYSIS. Suppose ACB the Δ required, having the sides AC, BC equal to the given ones. Produce the $\perp CD$ to meet the circle circumscribing the Δ at G , and draw $GP \parallel$ to AB to meet AC produced in P . Join AG, BG ; let CQ be a diam. of the circle, and $a : b$ the given ratio of the rectangle of the segments of the base to the square of the perpendicular. Then $AD \cdot DB = CD \cdot DG : CD^2 :: GD : CD :: GD \cdot CQ = AG \cdot GB : CD \cdot CQ = AC \cdot CB$ (Euc. Prop. c, VI.) $:: a : b$. Again by parallels $DG : CD (:: a : b) :: AP : AC$, therof. AP is given, wherof. $AG \cdot GB : AC \cdot CB :: AP \cdot BC : AC \cdot BC$, therof. $AG \cdot GB = AP \cdot BC = \text{a given rectangle}$, and $AG^2 - BG^2 = AD^2 - BD^2 = AC^2 - BC^2 = \text{a given space}$, and therof. AG and BG are given by the 87th Prop. of the Data. Now the locus of the point G is a semicircle, whose diam. is the given line CP , and A is a given point, and AG a given line, therefore the point G is given: Hence this

CONSTRUCTION. Produce the given side AC to P , so that $PA : AC :: a : b$, and on PC , as a diam. describe a semicircle PCG . Find, by Prop. 87 of the data, two lines AG, BG , so that their $\text{rectangle} = CB \cdot AP$, and the difference of their squares $= AC^2 - BC^2$, and from A , to the circle, apply AG (it being the greater or less of the two lines according as AC is the greater or less of the given sides); draw CG and perpendicular to it draw ADB , to which, from C , apply $CB = \text{the other given side}$, and it is done, as is manifest from the Analysis.

The same, answered by Mr. I. T. M'Donell.

ANALYSIS. Let the required $\triangle ACB$ be circumscribed by a circle, in which draw a diam. $EF \parallel$ to the \perp CD , which produce to meet the circle in G , and demit the \perp 's GH , FK ; produce FK to meet the circle in M and draw AM , AF . Now the rect. $ACB = CD \cdot EF$ is a given magnitude, also the ratio $CD^2 : AD \cdot DB$ or $CD \cdot DG = CD : DG$ is given; therefore $DG \cdot EF = HL \cdot EF = LF \cdot FE - HF \cdot FE = AF^2 - AM^2 = FK^2 - KM^2$, is a given magnitude. Again, CK is $= \frac{1}{2}$ the sum, and $AK = \frac{1}{2}$ the difference, of the given sides; therefore $AK \cdot KC = FK \cdot KM =$ a given rectangle. Whence the following

CONSTRUCTION. On any right line FM , take FK , KM of such magnitudes, that their rectangle may be equal to one fourth of the diff. of the squares of the sides, and the diff. of their squares to the rectangle of the sides in the given ratio of the rectangle of the segments of the base to the square of the perpendicular (which is a well known problem); then \perp to FM draw CKA , making $KC = \frac{1}{2}$ the sum, and $KA = \frac{1}{2}$ the diff. of the given sides, and having described a circle take $FB = FA$; draw AB , CB and it is done.

This question was elegantly constructed by Messrs. Cunliffe, Hill, Peacock, Swale, and Thornoby; and Mr. Barron sent an algebraical solution.

XV. QUESTION 197, answered by Mr. W. Wallace, Perth Academy.

Fig. 495, pl. 26. Let AB and CD be the straight lines given by position, take any points E and F in them and join EF , draw FG parallel to AB , draw EH perpendicular to the plane passing through FG and CD , meeting it in H , and draw HK parallel to FG , then HK is also parallel to AB , and since HE is perpendicular to the plane passing through FG , CD , a plane passing through EB , HK is perpendicular to the plane. Join EF , the triangle EHF is right angled at H , therefore $EF^2 = EH^2 + HF^2$; but since EH is a constant quantity, viz. the distance between the parallel lines AB , HK , the line EF will be the least possible when FH is the least possible. Therefore if HK , the common section of the planes EK , KD , be produced to meet CD in f and fe be drawn perpendicular to AB meeting it in e , it is evident that fe is the least distance between the given lines: for in this position FH vanishes and thus EF is perpendicular to both the given lines AB , CD .

All our other correspondents who answered this question referred to Emerson's Geometry.

XVI. QUESTION 198, answered by Mr. R. Simpson, *Croxdale, Durham.*

Let ES (fig. 492, pl. 26.) be the surface of the water, and GHLI the square aperture, each side of which is 15 inches. Now, it is shewn by the writers on Hydraulics (see Hutton's Course, vol. ii. pa. 226.) that the quantity flowing through ILHG is to that which would flow through an equal orifice placed horizontally at the depth EG, as the parabolic frustum IKHG is to ILHG. Put $a^2 = 15^2$, the area of the square hole, $g = 193$ inches, and $x = EG$, then $2a^2 \sqrt{gx}$ will represent the quantity of water discharged from the equal horizontal aperture in a second: and, by the nature of the parabola, we shall have $EG : GH^2 :: EL : IK^2$, that is $x : a^2 :: x - a : (a^2x - a^3) \div x$. Then $\frac{2}{3}ax =$ the area of the parab. EGH, and $\frac{2}{3}(x - a) \sqrt{(a^2x - a^3) \div x} =$ the area of the parab. EIK; the difference of these is the frustum

IKHG. Then say, as $a^2 : \frac{2}{3}ax - \frac{2x - 2a}{3} \sqrt{\frac{a^2x - a^3}{x}} :: 2a^2$

$\sqrt{gx} : 4\sqrt{gx} \frac{ax - x - a}{3} \sqrt{\frac{a^2x - a^3}{x}} =$ by the question to

166.2938889 gallons, or 46894.876653 cubic inches. After restoring the values of a and g in this equation, we soon, by trial and error, find $x = 64$ inches = EG, and consequently EI = 49 inches.

The same, answered by Merones Minor.

Put $g = 16\frac{1}{2}$, $b =$ the breadth or depth of the aperture, $h =$ height, or depth of the water above the top of the aperture, $x =$ any variable depth from the top of the water, and $q = 166.2938889$ gallons = $\frac{47}{288}q$ feet. Then, by Hutton's Course, vol. ii. pa. 223, art. 294, $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$, the velocity at the depth x : also $bx =$ the fluxion of the aperture, and $2bx \sqrt{gx} =$ the fluxion of the quantity, the correct fluent which, when $x =$

$h + b$, is $= \frac{4b}{3} (h + b)^{\frac{3}{2}} \sqrt{g} - h^{\frac{3}{2}} = \frac{47}{288}q$, by the question.

Hence $h = 4.071$, and $h + b = 5.321$ feet the depths required.

And thus the question was answered by Mr. James Cunliffe. All the other answers we received were false.

XVII. QUESTION 199, answered by Mr. John Blackwell.

ANALYSIS. Fig. 474, pl. 25. Let ACB be the required Δ , and $DEGF$ its inscribed rectangle whose length and breadth are given. Draw $DI \parallel$ to CB which will therefore be $= EB$. About the Δ ADI let a circle be described whose diam. is DK . Then the rect. $AD \cdot EB = AD \cdot DI = DF \cdot DK$, being given, the right line DK is given: Hence the following

CONSTRUCTION, by Merones Minor.

Upon any line DK , a 4th proportional to DF , DI and DA , as a diam. let a circle be described, from which cut off a segment to contain an angle $ADI =$ to the given vertical \angle , produce AI till $IB =$ the length of the given rectangle, and at a distance $=$ to the given breadth draw a right line \parallel to AB meeting the circle in D , on which take $DE = IB$; through D and E draw AC , BC which will meet in C the vertex of the required Δ .

The same, answered by Mr. John Lowry.

ANALYSIS. Suppose that ACB is the Δ required, and that $FDEG$ is the given inscribed rectangle. Draw $DI \parallel$ to CB to meet AB in I , then $DI = EB$, and $AD \cdot DI = AD \cdot EB =$ a given space. Draw DQ to make the $\angle IDQ = ADF$, and meet IQ drawn \perp to DI in Q ; then the Δ s ADF , DIQ , are sim. therf. $AD : DF :: DQ : DI$, or $DF \cdot DQ = AD \cdot DI =$ a given space, but DF is given, therf. DQ is given in magnitude, and it is also given by position; for, if to the $= \angle$ s IDQ , ADF , there be added the common $\angle FDI$, then $\angle FDQ = ADI =$ the given $\angle ACB$, and DF is given by position, therf. DQ is given by position. Again the locus of the point I is a semicircle whose diam. is DQ , and FG is given by position; therf. the point I is given. *Ergo Solutum.*

Let $FDEG$ be the given rectangle, draw DQ to make the $\angle FDQ =$ the given vertical \angle , and make $DQ \cdot DF =$ the given rectangle of the segments; on DQ , as a diam. describe a semicircle to intersect FG in I , join ID , and draw AD to make the $\angle ADI =$ the given one, and meet BEC , drawn through $E \parallel$ to ID , at C ; produce FG both ways to meet AC , CB at A and B , and it is done.

It is evident that the rectangle of the segments is a *minimum* when the semicircle touches FG in I .

This question was truly answered by Messrs. Cunliffe, Hill, May, M'Doneld, Swale, and Whalley.

XVIII. QUESTION 200, answered by Mr. John Lowry.

ANALYSIS. Fig. 475, Pl. 25. Let $\triangle ACB$ be the given \triangle , and $FGHE$ the inscribed rectangle, and let its perimeter be equal to a given line instead of a *maximum*. On AB demit the $\perp CD$, and let it meet the FG in R , then, by sim. $\triangle s$, $AB : CD :: FG : CR = CD - DR$, and by division $AB \propto CD : CD :: FG + DR \propto CD : CR$, but $FG + DR = FG + GH$ is given, being equal to half the perimeter, and AB and CD are given; theret. CR is given, and the construction is evident.

Now it is plain that $FG + DR$ increases as CR increases, therefore if AB be greater than CD , the perimeter will continue to increase till FG coincides with AB ; but, if CD be greater than AB , the perimeter will continue to increase till the points R, C , coincide; and when $CD = AB$ the perimeter will always be equal to the constant quantity $2AB$ or $2CD$. Wherefore it is evident, that the perimeter cannot be greater than twice the longest side of the \triangle nor less than twice the \perp demitted on that side from the opposite angle.

The same, answered by Mr. J. H. Swale.

GENERALLY. When, instead of a maximum, the perimeter is equal to a given line.

Demit, from the vertex C of the given $\triangle ACB$, the $\perp CD$, make $DQ = DB$, $QK = KA$, and join C, K . In DC produced take $DP = \frac{1}{2}$ the given perim.; at I , any point in DP , erect the $\perp IL = \frac{1}{2} IP$ and join PL , and produce it to meet CK at N . Through N , parallel to AB , draw FG , and demit upon AB the $\perp FE, GH$: the required rectangle is $EFGH$.

Join C, Q , and let CQ, FG meet FG, CD in T and R . By the constr. $AK = KQ$, and $QD = DB$, theret. $FN = NT$ and $TR = RG$; theret. $FT + TG = 2NT + 2TR = 2NR = FG$: But, since $PI = 2IL$, PR will be $= 2RN$, that is, $DR + 2RN = DR + RP = DP$. Now $DR = EF = GH$, $FG = EH$, therefore $EF + FG + GH + HE = 2EF + 2FG = 2DR + 2RP = 2DP =$ the given line, by construction.

When PL produced does not meet CQ , the problem becomes impossible, hence the utmost limit for the point N will be its coincidence with K , therefore FG will coincide with AB ; whence it appears that the perimeter cannot be greater than twice the longest side of the triangle.

Solutions were received from Messrs. Hill, May, McDonald, Merones Minor, and Peacock.

XIX. QUESTION 201, answered by Mr. James Cunliffe, Bolton

Analysis. Fig. 476, Pl. 25 Having described a figure and marked the angular points, &c. as in question 153, draw $MN \parallel$ to GC meeting CE in N . The \angle s MEN , HCG are obviously similar, therefore $HC : ME :: HG : MN = GC$, but HC and ME are homologous parts of the similar trapezia $ACBH$, and $DEFM$, and HG is the diameter of the circle circumscribing the trapezium $ACBH$, consequently GC must be $=$ to the diameter of the circle circumscribing the trapezium $DEFM$, or the Δ FED . Then because of the sim. Δ s ABC , FED , $HG : GC :: GC : (GC)^2 \div HG =$ the diameter of the circle circumscribing the Δ formed from FED ; and in like manner $(GC)^2 \div (HG)^2$ will express the diameter of the circle circumscribing the Δ formed from the preceding, &c. Therefore the expressions for the diameters of the several circles in succession will be

$$GC, \frac{(GC)^2}{HG}, \frac{(GC)^3}{(HG)^2}, \frac{(GC)^4}{(AG)^3}, \frac{(GC)^5}{(HG)^4} \&c.$$

which are evidently a series of lines in geometric proportion. Now it is well known that $(HG \cdot GC) \div (HG - GC)$ will express the sum of the said series continued *ad infinitum*, which put $= L$, the given line, or make $HG \cdot GC = L \times (HG - GC)$, whence $GC \times (L + HG) = L \times HG$, which points out the following method of determining the position of the vertex of the Δ ABC , viz. apply the chord $GC =$ a fourth proportional to $L + HG$, L , and HG .

The same, answered by Mr. Lowry.

Let MN be the diam. of the circle circumscribing the Δ EDF , drawn from M the middle of AB at right angles to the base DF and meeting it in m ; and from the points E , D , F , on DF and ME demit the \perp s Ee , Ff , and Dd , and draw mn parallel to GH to meet Ee , produced, in n . Produce CE to meet the circle $MEFD$ which, by reason of the right angle at E , it will do at N , the extremity of the diameter MN . Also on MN and EN , demit the perpendiculars EK , KL , LS , SQ , &c. alternately, and join GC . Then it is evident, from Prop. XV. Art. XXXI, of the Repository, that MN is equal and parallel to CG , and it may be shewn in the same way as in that proposition, that EN is equal and parallel to mn the diameter of the circle circumscribing the Δ formed by joining the points e , d , f , and that KN is equal and parallel to the diameter of the circle circumscribing the next Δ formed in a similar manner, LN equal and parallel to the next, SN

SN to the succeeding one, and so on. Hence, by sim. figures $HG : (GC) MN :: MN : EN :: EN : KN :: LN :: LN : SN$ and so on: theref. Euc. V. 12, $HG : MN$ as the sum of the antecedents $HG, MN, EN, KN, LN, \&c.$ to the sum of the consequents $MN, EN, KN, LN, \&c.$ that is, $HG : GC :: HG + \text{the given sum} : \text{the given sum}$, and by division $HG - GC : GC :: HG : \text{the given sum}$. *Ergo Solutum.*

Divide HG at Z so that $HZ : GZ :: HG : \text{the given sum}$, make $GC = GZ$, and C is the vertex of the Δ required.

COR. When $GC =$ the radius of the given circle, the sum of all the diameters is equal to twice GC .

Answers were also received from Messrs. Hill, May, M'Doneld, and Swale.

XX. QUESTION 202, answered by Mr. Cunliffe.

Let $SQVR$ (fig. 484, pl. 25.) be the given conic section, C its centre, and P the given point. Through P draw the diam. SV of the given curve, and upon P as a diam. describe another conic section similar to the given one and in a similar position, which will be the locus of the point of bisection, and the axes of this latter conic section will manifestly be parallel to those of the given curve. For, draw the right line PQR to cut the given curve in Q and R and intersect the other curve in m ; also through m draw the semidiameter CmT , and parallel to CT , through O the centre of the other curve, draw the semidiameter Ot , cutting PR in n . It is evident enough that Pm is bisected in n , therefore Pm is an ordinate to the semidiameter Ot , by the well known property of conic sections; and because the semidiameters Ot and CT are parallel and in curves similar and similarly situated it is a plain that QR will be an ordinate to the semidiameter CT , and will therefore be bisected by it in m . *Q. E. D.*

The above demonstration is founded on properties that are common to all the conic sections, and is therefore general, though the figure is only drawn for the ellipsis.

The same, answered by Mr. W. Wallace, Perth Academy.

Fig. 494, PL 26. Let P be the given point, EHF the given conic section, which any straight line drawn through P cuts in A and B ; let AB be bisected in D ; it is to be proved that the point D is also in a given conic section similar to EHF and similarly situated. Let C be the centre of the given conic section, join PC meeting the curve in E and F , draw the semi-diameter CDH , draw EG parallel to AB , meeting CH in K , and the curve in G and join FG . Because AB is bisected at D , it follows that AB is an

an ordinate to the semidiameter CH, and since EG is parallel to AB, EG is also an ordinate to the same semidiameter, and is therefore bisected at K; now the diameter EF is bisected at C, therefore CK or CD is parallel to FG; thus it appears that the triangles EGF, PDC are similar, and similarly situated: and since EF, PC, homologous sides of these triangles, are given in position and magnitude, and EFG one of them, is inscribed in a given conic section EHF, the other triangle PDC is also inscribed in a given conic section similar to EHF, and similarly situated; therefore the locus of D is a given conic section, &c. as was to be demonstrated.

XXI. QUESTION 203, answered by Mr. James Whalley.

ANALYSIS. Fig. 477, Pl. 25. Let the figure be constructed by the given data, and produce AD to meet BG, draw \parallel to OD, in G, and join CD. Then, because $\angle FDE = FCD$, and $\angle F$ common to the \triangle s FDE, and FCD, these \triangle s are similar, *theref.* $FE : FD :: FD : FC$, and, by the parallel lines OD, BG, $FD^2 : FE^2 :: BG^2 : BE^2$, *theref.* $FE \cdot FC : FE^2$, that is, $FC : FE :: BG^2 : BE^2$: but the ratio of $FC : FE$ is given, and BG is given, being equal to the given diam. AB; *theref.* BE will be given: Hence the following

CONSTRUCTION, by Mr. James Cunliffe.

Let the given ratio of CE to EF be denoted by that of m to n . Apply BE to AD, such that $m + n : n :: BA^2 : BE^2$, and produce BE to meet the circle at C, and the thing is done, as is evident from the analysis.

From the preceding analysis the following theorem is evident. If two chords AD and BC, drawn from the extremes A and B of the diameter of a circle, intersect in E, and the radius OD be drawn cutting BC in F; then will $FC : FE :: AB^2 : BE^2$.

The same, answered by Mr. J. H. Swale.

CONSTRUCTION. Divide AD at G' in the given ratio, and apply, to AB, DP, the side of a square equal to the rect. $AD \cdot DG'$; draw DC, making the $\angle ODC = DPB$, to meet the circle in C and join BC meeting DA, DO, in E and F and it is done.

DEMONSTRATION. The \triangle s DFC, DFE are sim. for the $\angle DCF = DAB = EDF$, and the $\angle CFD$ and the side DF are common to both, *theref.* $CF : FD :: FD : FE$, and *theref.* $CF \cdot FE = FD^2$. The \triangle s DFC, PDA are also sim. for the $\angle CDF = DPA$, by constr. *theref.* $CF^2 : FD^2 = AD^2 : DP^2$, that is, $CF^2 : FC \cdot FE = AD^2 : AD \cdot DG'$, or, $CF : FE = AD : DG'$, *theref.* dividendo, $CE : EF = AG' : G'D =$ the given ratio by Constr.

When the ratio is that of equality, bisect AD in G'.

The

The same, answered by Mr. I. T. McDonell.

CONSTRUCTION. Let ADB be the given semicircle, and AD the chord given by position. Draw BG \parallel to the radius DO meeting AD produced in G, and through the points A, B, G describe a circle. Also in BD produced take DH : DB equal to the given ratio, and draw HI \parallel to DA meeting the circle at I; join BI, cutting the given semi-circle at C, the given chord in E, and the radius in F, so shall BC be the line required to be drawn.

DEMONSTRATION. Demit, upon AG, the \perp IL, and join CA, CL. Then, because ICA, ILA are right angles, the points A, C, L, I, are in a circle, also $\angle ADO = AGB = AIB = \text{supp. of } ALC = CLE$; theref. CL, DF are parallel; wheref. by sim. Δ s, HD : DB = IE : EB = LE : ED = CL : DF = CE : EF = the given ratio.

COR. If BH be produced to meet the circle at K, BK will be the diameter, for it bisects the chord AG perpendicularly : moreover it appears that the ratio of CE to EF will always be less than that of KD to DB.

The same, answered by Mr. Louis Hill.

CONSTRUCTION. Let $m : n$ be the given ratio of CE \parallel EF, and cut AC at H', so that AH' : H'O :: $m + n : n$; draw HT' \perp to AO to meet a semicircle described on AO at I', and draw AI' to meet OK', drawn \perp to AO, at K'. From O, to the chord AD, apply OR = OK' and draw BFC \parallel to OR and it is done.

For, by constr. Eu. cor. 8. VI, and sim. Δ s, AO² (DO²) : OK² (OR²) :: AI' : IK' :: AH' : H'O :: $m + n : n$; but by sim. Δ s, DF : FE :: DO : OR, or, DF² : FE² :: DO² : OR² :: $m + n : n$. Again, since OD = OA, $\angle ADO$ is = BAD = BCD (Euc. III. 21.); theref. the Δ s FCD, EDF are similar; wherefore DF : EF :: CF : DF, theref. DF² = EF \cdot CF, and theref. EF \cdot CF : EF² :: CF : EF :: $m + n : n$, theref. by division, CE : FE :: $m : n$, that is, in the given ratio.

Mr. Lowry also answered this question.

XXII. QUESTION 204, answered by Mr. J. H. Swale.

Fig. 478. Pl. 25. Construction. Divide the base AB in the given ratio at T, and demit, from D, upon AB, the \perp DP, in which take PS : PD = BT : BO, the given line; draw SH \parallel to AB to meet AC in H, join BH meeting DE in G, and make GF = BO, and join BF meeting AC in I, and it is done.

Demon-

Demonstration. Demit upon AB the \perp s HL, GN, and produce SH to meet BI at K and join HT. Then, since $FG = BO =$ the given line, by constr. it only remains to prove that $AH : HI =$ the given ratio. Now by constr. $PS = LH : PD = NG :: BT : BO = FG ::$, per parallels, $KH : FG$, that is, $HK = BT$: but HK, BT are also parallel, therof. BI, TH are parallel. Consequently $AT : TB = AH : HI =$ the given ratio, by constr.

The same, answered by Mr. Lowry.

Analysis. Suppose the problem solved. Draw $IK' \parallel$ to BH to meet AB produced in K' and DE in M. Then, by parallels, $AB : BK' :: AH : HI$, i.e. a given ratio, but AB is given in magnitude, therof. BK' is given, and consequently GM is given. Also, by parallels, $BK' : FM :: AB : DF$, therof. DF is given. *Ergo Solutum.*

On AB produced take $BK' : AB$ in the given ratio of HI to AH, and take $BL' =$ the given length of GF; on DE take DF so that $DF : AB :: LK' : BK'$, and draw BF to meet AC in I and join FL' , then if BGH be drawn \parallel to FL' it is done, as is evident from the analysis.

The same, answered by Mr. James Whalley.

Fig. 479. Pl. 25. *Analysis.* Through F draw $Fn \parallel$ to AC meeting BH, and AB in l and n , also draw $Bk \parallel$ to AC meeting DE in k . The Δ s Bln , and G/F are sim. whence $Fl : ln :: GF : Bn = Fk$, and, by the parallels Fn , and AC, $Fl : ln :: HI : AH$, therof. $HI : AH :: GF : Fk$; but GF is given, and the ratio of HI : AH is also given, therof. Fk is given; hence the constr. is evident.

Elegant constructions were also received from Messrs. Cunliffe, Hill, and M'Doneld.

XXIII. QUESTION 205, answered by A.B. the proposer.

Because the n th differences of a, b, c , &c. are equal, by the differential method, the series $= a + (a + d')x + (a + 2d' + d'')x^2 + \&c.$ Let the required scale $= Ax^{n+1} + Bx^n + Cx^{n-1} + \&c. + Gx^4 + Hx^3 + Ix^2 + Kx$, then by the nature of the problem we shall have,

$$\begin{aligned}
 & a \times Ax^{n+1} \\
 & + (a + d')x \times Bx^n \\
 & + (a + 2d' + d'')x^2 \times Cx^{n-1} \\
 & \quad \&c. \quad \quad \quad \&c.
 \end{aligned}$$

$$\begin{aligned}
& \left\{ a + (n-3)d + \frac{(n-3)(n-4)}{1 \cdot 2} d^2 + \frac{(n-3)(n-4)(n-5)}{1 \cdot 2 \cdot 3} d^3 + 8c \right\} x^{n-3} \times Gx^4 \\
& + (n-3) d^{n-3} \\
& \left\{ a + (n-2)d + \frac{(n-2)(n-3)}{1 \cdot 2} d^2 + \frac{(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3} d^3 + 8c \right\} x^{n-2} \times Hx^3 \\
& + (n-2) d^{n-2} \\
& \left\{ a + (n-1)d + \frac{(n-1)(n-2)}{1 \cdot 2} d^2 + \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} d^3 + 8c \right\} x^{n-1} \times Ix^2 \\
& + (n-1) d^{n-1} \\
& \left\{ a + nd + \frac{n(n-1)}{1 \cdot 2} d^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} d^3 + 8c \right\} x^n \times Kx \\
& + \frac{(n-3)(n-2)(n-1)}{4 \cdot 3 \cdot 2 \cdot 1} d^4 + \frac{(n-2)(n-1)n}{3 \cdot 2 \cdot 1} d^{n-3} + \frac{(n-1)n}{2 \cdot 1} d^{n-2} + d^{n-1} + d^n \\
& \left\{ a + (n-1)d + \frac{(n+1)n}{1 \cdot 2} d^2 + \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3} d^3 + 8c \right\} x^{n+1} \\
& + \frac{(n-3)(n-2)(n-1)}{6 \cdot 4 \cdot 3 \cdot 2 \cdot 1} d^5 + \frac{(n-2)(n-1)n}{4 \cdot 3 \cdot 2 \cdot 1} d^{n-3} + \frac{(n-1)n(n+1)}{3 \cdot 2 \cdot 1} d^{n-2} + \frac{n(n+1)}{2 \cdot 1} d^{n-1} + (n+1)d^n
\end{aligned}$$

which is the next term in the series. The above will be evident from the nature of recurring series. By comparing the coefficients

of the same differences it will appear that $K = n + 1$; $I + nK = \frac{n(n+1)}{2 \cdot 1}$, therefore $I = \frac{n(n+1)}{1 \cdot 2} - n(n+1) = -$

$$\frac{n(n+1)}{1 \cdot 2}; H + (n-1)I + \frac{(n-1)n}{1 \cdot 2}K = \frac{(n-1)n(n+1)}{1 \cdot 2 \cdot 3},$$

therefore $H = (n-1) \left(\frac{n(n+1)}{1 \cdot 2 \cdot 3} - \frac{n}{2}K - I \right) = (n-1)$

$$\left(\frac{n(n+1)}{2 \cdot 3} - \frac{n(n+1)}{1 \cdot 2} + \frac{n(n+1)}{1 \cdot 2} \right) = \frac{(n-1)n(n+1)}{1 \cdot 2 \cdot 3},$$

$$G + (n+2)H + \frac{(n-1)(n-2)}{1 \cdot 2}I + \frac{(n-2)(n-1)n}{1 \cdot 2 \cdot 3}K =$$

$$K = \frac{(n-2)(n-1)n(n+1)}{1 \cdot 2 \cdot 3 \cdot 4}, \text{ therefore by transposition \&c.}$$

$$G = -\frac{n-2}{1 \cdot 2 \cdot 3 \cdot 4} \frac{(n-1)n(n+1)}{4}, \text{ \&c. \&c. Wherefore, } Kx +$$

$$Ix^2 + Hx^3 + Gx^4 + \&c. \dots \dots Cx^{n-1} + Bx^n + Ax^{n+1} =$$

$$(n+1)x - \frac{(n+1)n}{1 \cdot 2}x^2 + \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3}x^3 -$$

$$\frac{(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4}x^4 - \&c. = 1 - (1-x)^{n+1} =$$

scale of relation required.

Hence it is evident that if the n th differences be equal, the number of terms in the scale of relation $= n + 1$.

Mr. Cunliffe also answered this question.

XXIV. QUESTION 206, answered by Mr. James Cunliffe.

Fig. 480, Pl. 25. By the hyp. $\angle DAF = DAB$, also $AF = AB$, and the side AD is common to both the \triangle s AFD and ABD , therf. Eu. I. 4, $FD = BD$, and $\angle AFD = ABD =$ a right \angle . But FG is a tangent to the arc FB at F , wherefore the $\angle AFG$ is a right \angle (Euc. III. 18), for which reason FG will pass through D .

Through e , the middle of FD , draw the line Ah , to meet Ch , drawn \perp to AC , in h , and intersect DE in g . The \triangle s AED , and AeD , are by the conf. mutually equal in every respect, therefore the \triangle s ADG , and ADG , are also equal in every respect, and therefore $Bg = FG$.

Because AD bisects the $\angle BAC$; by Eu. VI. 3,

AC

$AC : AB = AF :: DC : BD$, and by the parallels De , Cd ,
 $Cd : De = Fe :: gC : Dg$; but
 $Cd : Fe :: AC : AF :: DC : BD$; therefore
 $DC : BD :: gC : Dg$, whence, by composition,
 $BC : BD :: DC : Dg$, that is, $BC : DC : BD : DG$. Now
 $AC : AB :: DC : BD$, as shewn above, whence, Eu. VI. 16,
 $BD \times AC = DC \times AB$; and by Simpson's Geom. III. 26,
 $AC \times AB - DC \times BD = AD^2$, whence
 $BD \times AC \times AB - DC \times BD^2 = BD \times AD^2$, that is,
 $DC \times AB^2 - DC \times BD^2 = BD \times AD^2$, that is
 $DC \times (AB^2 - BD^2) = BD \times (AB^2 + BD^2)$. Therefore
 $AB^2 - BD^2 : AB^2 + BD^2 :: BD : DC$, and by composition,
 $2AB^2 : AB^2 + BD^2 :: BC : DC :: BD : DG$, as was shewn before,
 $2AB^2 \times DG = BD \times (AB^2 + BD^2)$, that is*
 $4AB^2 \times DG = BD \times (2AB^2 + 2BD^2)$; but
 $4AB^2 \times FD = BD \times 4AB^2$, therefore, by adding,
 $4AB^2 \times FG = BD \times (6AB^2 + BD^2) = 6BD \times AB^2 + 2BD^3$. But
 $(AB + BD)^3 - (AB - BD)^3 = 6BD \times AB^2 + 2BD^3$, consequently
 $4AB^2 \times FG = (AB + BD)^3 - (AB - BD)^3$. Q. E. D.
 COR. $AB^2 : AD^2 :: DE : DG$. See the expression above,
 marked *.

The same, answered by Mr. John Lowry.

Let the line joining the points F and B , intersect AD at H ,
 then, since AD bisects the $\angle BAE$, FB will be \perp to AD , and
 $BH = HF$. Again, since DB and FG are tangents to the cir-
 cle at B and F , and $\angle BAD = DAF$, it is evident that FG
 passes through the point D , and that ED is $= DB$, therefore,
 since, $DE = EB$, and $FH = HB$, the straight line which joins the
 points H and E , will be parallel to FG , and the $\angle EHD =$
 $HDF = EDH$, therf. EH is $=$ to ED . Wherefore, by sim. Δs ,
 $DG : HE :: AD : AH :: AD^2 : AD \cdot AH = AB^2$ (Eu. VI. 8.), or
 $2DG : BD :: AD^2 = AB^2 + BD^2 : AB^2$, therefore
 $BD \times (AB^2 + BD^2) = 2DG \times AB^2$, or
 $2BD \times (AB^2 + BD^2) = 4DG \times AB^2$; but
 $4DB \times AB^2 = 4DG \times AB^2$; therf. by addition,
 $6 \times BD \times AB^2 + 2BD^3 = 4FG \times AB^2$. But it is evident
 from Proposition E, Art. XXII. of the Repository that,
 $6 \times BD \times AB^2 + 2BD^3 = (AB + BD)^3 - (AB - BD)^3$; therf.
 $4 \times FG \times AB^2 = (AB + BD)^3 - (AB - BD)^3$. Q. E. D.

*The same, answered by Mr. Richard Nicholson, Private Teacher
 of the Mathematics, at Liverpool.*

LEMMA. The cube on the sum of two lines less the cube on
 their difference is equal to that solid, the base of which is six
times
2 Q 2

times the square of the greater and the height the less, together with twice the cube on the less. For,

$$(AB+BD)^3 = AB^3 + 3AB^2 \cdot BD + 3AB \cdot BD^2 + BD^3, \text{ and} \\ (AB-BD)^3 = AB^3 - 3AB^2 \cdot BD + 3AB \cdot BD^2 - BD^3; \text{ therefore} \\ (AB+BD)^3 - (AB-BD)^3 = 6AB^2 \cdot BD + 2BD^3.$$

This being premised we have only to prove that

$$4AB^2 \cdot FG = 6AB^2 \cdot BD + 2BD^3.$$

DEMONSTRATION. By Euc. I. 26, FG intersects BC in D, and FD is = BD. Perpendicular to AD draw BH, and join HE, then will HE = BE = BD, therof. Euc. I. 32, twice the \angle HBE (=BAE) = HED, and, by Simpson's Geometry III. 19, and Euc. III. 22, the \angle BAE (=HED) = EDG; therof. HE is \parallel to DG, consequently

$$AD : AH (:: AD^2 : AD \cdot AH = AB^2) :: DG : DE; \text{ hence} \\ 2AD^2 \cdot 2DE (= 2AB^2 \cdot BD + 2BD^3) = 4AB^2 \cdot DG,$$

and by adding $4AB^2 \cdot DB$, we have

$$6AB^2 \cdot BD + 2BD^3 = 4AB^2 \cdot FG.$$

Q. E. D.

Demonstrations were likewise received from Messrs. Hill, M'Doneld, Swale, and Thornoby.

XXV. QUESTION 207, answered by Miss May.

To render the solution of this question as plain as possible, it will be necessary to point out the method of graduating a thermometer according to Fahrenheit's scale, which is thus: the tube being fixed to a frame, upon which the graduations are to be made, is put into water just freezing, and against the surface is marked 32; the tube being then put into boiling water and against the surface of the fluid is marked 212. Then the interval between these two marks being divided into 180 equal parts and marked 33, 34, &c.—55 will be temperate, 76 summer heat, and 98 blood heat. Now in Mr. Bulmer's thermometer the interval between freezing and boiling is divided into 205 equal parts. Therefore it is evident that

$$\begin{array}{rcl} 55 - 32 & & 26\frac{7}{8} \\ 180 : 76 - 32 :: 205 : 50\frac{1}{2} \\ 98 - 32 & & 75\frac{1}{8} \end{array}$$

$$\begin{array}{rcl} \text{Hence } 25 + 26\frac{7}{8} & = & 51\frac{7}{8} \text{ Temperate} \\ 25 + 50\frac{1}{2} & = & 75\frac{1}{2} \text{ Summer Heat} \\ 25 + 75\frac{1}{8} & = & 100\frac{1}{8} \text{ Blood Heat.} \end{array}$$

Ingenuous solutions to this question were received from Messrs. Johnson, Hill, Lowry, M'Doneld, Merones Minor, and Nicholson.

XXVI. QUESTION 208, answered by the proposer A. B.

Fig. 481, 482, Pl. 25. Let PQA be the orbit described by the body, F, the centre of force, FY perpendicular to the tangent PY,

PY, and Fy, to Qy, FQ being drawn indefinitely near to EP. Let S be the true place of the star, SA and Sa the aberrations belonging to the points P and Q; then it is well known that SA, Sa, are respectively parallel to PY, Qy, and proportional to $\frac{1}{FY}$ and

$\frac{1}{Fy}$; hence the angle YPy = YFy = ASa. Put SA = Y and

ST = P, then in the curve PQA, QF = $\frac{FY \times PT}{PY} =$

$\frac{p\dot{y}}{\sqrt{(y^2 - p^2)}}$, therefore the angular velocity of PF = $\frac{p\dot{y}}{y\sqrt{(y^2 - p^2)}}$

$= \frac{y^{n-1}\dot{y}}{\sqrt{(a^2y^2 - y^{2n})}}$. But $\frac{\dot{y}}{y} : \frac{\dot{p}}{p} :: \left(\frac{\dot{y}}{y} : \frac{n\dot{y}}{y}\right) 1 : n ::$

$\frac{y^{n-1}\dot{y}}{\sqrt{(a^2y^2 - y^{2n})}} : \frac{ny^{n-1}\dot{y}}{\sqrt{(a^2y^2 - y^{2n})}} = \text{angular velocity of SA} =$

$\frac{PY}{Y\sqrt{(Y^2 - P^2)}}$. Because SA = Y $\propto \frac{1}{p} \propto \frac{a}{y^n}$, let Y = $\frac{ma}{y^n}$,

m being constant, then $y = \frac{(ma)^{\frac{1}{n}}}{Y^{\frac{1}{n}}}$ and $\dot{y} = -$

$\frac{1}{nY} \frac{(ma)^{\frac{1}{n}} \dot{Y}}{Y^{\frac{1}{n}}}$. Hence by substitution and reduction $\frac{PY}{Y\sqrt{(Y^2 - P^2)}}$

$= - \frac{\dot{Y}}{\sqrt{(a^2 \times ma^{\frac{2-2n}{n}} \times Y^{\frac{4n-2}{n}} - Y^2)}}$, therefore $\frac{P}{Y^2 - P^2 Y^2} =$

$\frac{1}{a^{\frac{2}{n}} \times m^{\frac{2-2n}{n}} \times Y^{\frac{4n-2}{n}} - Y^2}$, and $P^2 = \frac{Y^{\frac{2}{n}}}{a^{\frac{2}{n}} \times m^{\frac{2-2n}{n}}}$, there-
fore

for: $P = \frac{\frac{1}{Y^n}}{\frac{1}{a^n} \times m^{\frac{1-n}{n}}}$, the equation of the curve required.

If $n = \frac{1}{2}$, then $P = Y^2 \div a^m$, an equation to a circle, the point S being in the circumference. If, therefore, a body move in a parabola, the centre of force being in the focus, the aberration curve will be a circle, the true place of the star being in the circumference.

If a body move in a circle and the centre of force in the circumference, then $n = 2$, and $P = \frac{Y^{\frac{1}{2}}}{a^{\frac{1}{2}} \times m^{-\frac{1}{2}}} = \frac{(m)^{\frac{1}{2}}}{a} \times Y^{\frac{1}{2}}$, an equation to the parabola.

If $n = 1$, then the curve PQA is the log. spiral, and $P = Y \div a$, or $P : Y :: 1 : a$, therefore the aberration curve is a log. spiral similar to the other.

The same, answered by Mr. Lowry.

Fig. 483, Pl. 25. Let NBb represent the curve in which the body is supposed to revolve, C the centre of force, B the place of the body at any assigned time, and S that of the star. Draw the tangent PAB and take BA to BS as the velocity of the body when at B is to the velocity of light. Draw SQ equal and parallel to BA; then from the nature of aberration (See Simpson's Essays, Prop. 1 and 2,) it is plain, that Q will be the apparent place of the star when the body is at B. Again, suppose the body to be at b, an indefinitely small distance from B, and that q is the apparent place of the star when the body is in that situation. Draw the tangent bp to meet CP, drawn perpendicular BP, at p, join Bb, and CB demit the perpendicular bb'; join also Sq, Qq, and draw QD perpendicular to Sq. Let IQ be a tangent to the curve of aberration at Q, and SI a perpendicular to that tangent. Then it is evident, from the propositions above cited, that Sq is parallel to bp, and therefore the triangles BPp, SQD, are similar; moreover Bp is ultimately equal to BP, and SQ to SD, wherefore BP (Bp) : Pp :: SQ : QD, and permutando, BP : SQ :: Pp : QD, therefore BP : bb' : SQ : bb' :: Pp : CB : QD : CB.

Again, the triangles CBP, bBb' are similar, therefore PC : PB :: bb' : Bb', or PC · Bb' = PB · bb', consequently Pp · CB : PB · bb' :: Pp · CB : PC · Bb', therefore by permutation and equality, SQ · bb' : QD · CB :: PC · Bb' : Pp · CB.

But,

But, by the question, CP is $= \frac{(CB)^n}{a}$, therefore its fluxion Pp is $= \frac{n}{a} \cdot (CB)^{n-1} \cdot Bb'$, hence $Pp \cdot CB = \frac{n}{a} \cdot (CB)^n \cdot Bb'$, and therefore $Pp \cdot CB = n \cdot PC \cdot Bb'$, and $QD \cdot CB = n \cdot SQ \cdot bb'$, or $QD^a \cdot CB^a = n^a \cdot SQ^a \cdot bb'^a$. Again, from what is done above and Euc. I. 47, $bb'^a = \frac{PC^a \cdot Pb'^a}{CB^a - PC^a}$, and from the similar triangles SQI , QqD , $QD^a = \frac{SI^a \cdot Dq^a}{QS^a - SI^a}$; these values of bb'^a and QD^a being substituted in the above expression, it becomes $\frac{SI^a \cdot Dq^a \cdot CB^a}{QS^a - SI^a} = \frac{n^a \cdot PC^a \cdot Bb'^a \cdot SQ^a}{CB^a - PC^a}$, from which equation, the relation of SI to SQ may be easily determined. For, by Prop. XI. Cor. 2, Emerson on Centripetal Forces; the velocity of the body in the curve at B is reciprocally as the perpendicular CP , therefore SQ (BA) will be reciprocally as CP or $\frac{a}{CB^n}$. Now let

$SQ : \frac{a}{SB^n}$ be assumed as b to 1; then CP is $= \frac{b}{SQ}$, $SQ =$

$\frac{ab}{CB^n}$, and $CB^a = \left(\frac{ab}{SQ}\right)^{\frac{2}{n}}$; therefore the fluxion of CB squared,

or $Bb'^a = \frac{(ab)^{\frac{2}{n}}}{n^a} \cdot \frac{DQ^a}{\frac{2n+2}{n}}$, and these values of PC^a , CB^a , and

Bb'^a being substituted in the preceding equation, it becomes, after

making the necessary reduction, $\frac{SI^a}{QS^a - SI^a} = \frac{SQ^{\frac{2}{n}}}{a^{\frac{2}{n}} \cdot b^{\frac{2-2n}{n}} \cdot (QS^a - SI^a)}$,
from

from whence it is evident $SI = \frac{SQ^{\frac{1}{n}}}{a^{\frac{1}{n}} \cdot b^{\frac{1-n}{n}}}$ the equation of the re-

quired curve.

XXVII. QUESTION 209, answered by Mr. James Cunliffe.

ANALYSIS. Fig. 485, Pl. 26. Suppose the thing done. To the centre O of the circle EF draw the line AO, and towards the centre S of the circle BC, take $\angle OAO = FAC$, the given angle, and $Ao = AO$. With the centre o describe another circle equal to EF, and produce AC, if necessary, to cut the periphery of this circle in e and f. Draw the lines AS, Ao, the radii SB, oe, and upon AC let fall the \perp s Sz, am; also draw OM \perp to AF; take away the $\angle OAC$ from the equal \angle s OAO, FAf, and there will remain the $\angle oAm = OAM$, theref. by Euc. I. 26, $om = OM$, and $ef = EF$, by Euc. III. 14.

CASE I. When the sum of the squares of the chords EF and BC is given.

By Euc. I. 47, $em^2 + mo^2 = oe^2$, and $Bn^2 + nS^2 = SB^2$, theref. $em^2 + Bn^2 + mo^2 + Sn^2 = oe^2 + SB^2 =$ a given magnitude: But $em^2 + Bn^2 = \frac{1}{4}(ef)^2 + \frac{1}{4}(BC)^2 = \frac{1}{4}(EF)^2 + \frac{1}{4}(BC)^2$ is = to a given magnitude, by the question; therefore $mo^2 + Sn^2$ is given.

Again, by Euc. I. 47, $Am^2 + mo^2 = Ao^2$, and $An^2 + Sn^2 = AS^2$, theref. $Am^2 + An^2 + mo^2 + Sn^2 = Ao^2 + AS^2 =$ a given magnitude: But $mo^2 + Sn^2$ has been shewn to be given, therefore $Am^2 + An^2$ is given. Now the loci of the points m and n, are semicircles intersecting in A, whose diameters are Ao and AS; therefore this case of the problem is reduced to question 783, Gentleman's Diary.

CASE II. When the difference of the squares of FF and BC is given.

$em^2 + mo^2 = oe^2$, and $Bn^2 + Sn^2 = SB^2$; and taking the diff. $em^2 - Bn^2 + mo^2 - Sn^2 = oe^2 - SB^2 =$ to a given magnitude. But $em^2 - Bn^2 = \frac{1}{4}(ef)^2 - \frac{1}{4}(BC)^2 = \frac{1}{4}(EF)^2 - \frac{1}{4}(BC)^2$ is a given magnitude, by the question, therefore $mo^2 - Sn^2$ is given. Again

$Am^2 + mo^2 = Ao^2$, and $An^2 + Sn^2 = AS^2$; taking the diff. $Am^2 - An^2 + mo^2 - Sn^2 = Ao^2 - AS^2 =$ a given magnitude: but $mo^2 - Sn^2$ has been shewn to be given, therefore $Am^2 - An^2$ is given; therefore this case of the problem is reduced to question 803, Gentleman's Diary.

And thus the question was answered by Mr. James Whalley.

The

The same, answered by Mr. Lowry.

ANALYSIS. Fig. 486. Pl. 26. Let O and Q be the centres of the given circles, and on the chords CB, EF demit the perpendiculars OI, QH, and produce OI to meet EF at G, draw the radii OB, EQ, and join AO, AQ.

Then, because the sum or difference of the squares of BC, EF is given, the sum or difference of the squares of their halves IB, EH will be given: but $AO^2 - BO^2 = AI^2 - BI^2$, and $AQ^2 - EQ^2 = AH^2 - EH^2$, therefore the sum or difference of the squares of AI, AH is given, and because IAG is a given \angle and GI is \perp to AI, the \angle AGI is given, and the ratio of AG to AI is also given. Therefore the sum or difference of the square of AH and the space to which the square of AG has a given ratio is given. Through the points A and O, describe a segment of a circle AGO to contain an \angle equal to the given one AGI, this circle is given by position. Let AK be its diameter, and join KG, also on AG take $AR = AI$, and draw $RL \parallel$ to GK to meet the diameter AK in L. Then by sim. Δ s, $AG : AR = AI :: IK : AL$; but AK, and the ratio of $AG : AI$ are given, therefore AL is given, and L is a given point. Now the loci of the points H and R are the semicircles QHA and LRA, described on the diameters AQ, AL, respectively. Hence, the problem is reduced to this, viz. To draw AH so that the sum or difference of the squares of the chords AR, AH, may be given, which is a particular case of the general problem solved on page 362 of the Repository.

The same, answered by Mr. Richard Nicholson, Liverpool.

ANALYSIS. Fig. 487, Pl. 26. Draw AD, AG to the centres D and G of the given circles, upon the diameters AD, DG describe the circles ADI, AGH intersecting each other in E, and cutting AF, AC in I and H. Join B, I; G, H; D, F; G, C; L, I; L, H; and \perp to LI, LH draw AK, AP; join KP, and upon the diameter AL describe the circle AKLP, which will pass through the points K and P, because the \angle s AKL, APL are right ones. Now, the \angle s AID, AHG, standing in semicircles are right angles, therf. $AD^2 - DI^2 = AI^2 - IF^2$, and $AG^2 - GH^2 = AH^2 - HC^2$, consequently, by the addition of equal spaces, we have $AD^2 + AG^2 + IF^2 + HC^2 - DI^2 - GH^2 = AI^2 + AH^2$, a given space, for $IF^2 + HC^2$ is $= \frac{1}{4}$ th of the given space $EF^2 + BC^2$, and AD^2, AG^2, DI^2, GH^2 are squares upon given lines: Now, the \angle s AIL, AHL, standing upon the given chord AL of the given circles AIL, AHL, are given, therf. the right

right angled Δ s AIK, and APH are given in specie, and since the angle FAC is given by the problem, the \angle KAP is also given, and thence KP, being the chord on which the angle KAP stands in the given circle AKLP; moreover, the ratio of $AI^2 : AK^2$ is given, which suppose as $m : n$, and the ratio of $AH^2 : AP^2$ is also given, which suppose as $r : n$; then $m \cdot AK^2 + r \cdot AP^2 = n \cdot (AI^2 + AH^2)$ a given solid; whence the Δ AKP is easily determined by the Lemma on page 174, of Simpson's Select Exercises. Hence the position of the required lines becomes known.

When the diff. of the squares of BC and EF is given, the diff. of $m \cdot AK^2$ and $r \cdot AP^2$ is given, in which case, as well as the former, the locus of the vertex, A, of the Δ APK is a circle, which is proved in my solution to the 24th question in the Mathematical Companion.

The same, answered by Mr. J. H. Swale.

CASE I. For the sum of the squares. Fig. 488, Pl. 26.

CONSTRUCTION. Let O and Q be the centres of the given circles and UW half the side of a square = the given space. Join AO, AQ, and draw AH = AO, making \angle OAH of the given magnitude, and produce QA to I, making AI = AQ; join IH, upon which as a diameter let a circle be described, and let HK be taken = a quadrant thereof. Join AK, and let a circular segment be described thereon, capable of containing an angle = to the supplement of half a right angle, to which apply HP, the side of a square = the difference between the sum of the squares of UW, AO, AQ, and OC, QF. Join H and S, the point where KP, produced, meets the circle on IH; draw AEF \parallel to SH, and ABC making the \angle FAC = HAO, and it is done.

DEMONSTRATION. Join OB, OC, QE, QF, and demit upon BC, EF the \perp s OD, OG, HL, join also PA, IS, and produce PA, SI to meet HS, FA produced, at T and N. Now by the circle's property

$FQ = QE$, $FG = GE$, $OC = OB$, and $CD = DB$, also, $AQ^2 - QF^2 = AG^2 - GF^2$, and $AO^2 - OC^2 = AD^2 - DC^2$, that is, $FG^2 = AG^2 + QF^2 - AQ^2$, and $CD^2 = AD^2 + OC^2 - AO^2$; therefore $FG^2 + DC^2 = AG^2 + AD^2 + QF^2 + OC^2 - AQ^2 - AO^2$.

Again, by constr. the \angle s TPS, TSP are each = half a right \angle , and PTS = a right angle; therefore, by Eu. I. 47, $HT^2 + TP^2 = HP^2 = UW^2 + AO^2 + AQ^2 - OC^2 - QF^2$, by constr.

Again, by reason of = \angle s and parallels, $PT = TS = AN =$ also to AG, for AI = IQ; and $HT = LA$; then $HT^2 + TP^2 = HT^2 + TS^2 = LA^2 + AG^2 = UW^2 + AO^2 + AQ^2 - OC^2 - QF^2$.

But,

But, by constr. $\angle HAO = FAC$, or $\angle HAL = OAD$, and $AH = AO$, that is, $AL = AD$. Hence,

$$AG^2 + AL^2 = AG^2 + AD^2 = UW^2 + AO^2 + AQ^2 - OC^2 - QF; \text{ but}$$

$$AG^2 + AD^2 = FG^2 + CD^2 + AQ^2 + AO^2 - OC^2 - QF^2; \text{ therf. } FG^2 + CD^2 = UW^2, \text{ or } 4FG^2 + 4CD^2 = EF^2 + BC^2 = 4UW^2, \text{ equal the given space by the construction. } Q. E. D.$$

The *maximum* will obtain when HP passes through O' the centre of the circle described on AK; the *minimum* will also be determined in the same circumstance, only it will be the opposite point of the circle to P.

CASE II. For the difference of the squares.

CONSTRUCTION. Let O, Q, be the centres as in the first case, and YZ half the side of a square = the given space. Join AO, AQ, and draw AH = AO, making the $\angle OAH =$ the given one. Join HQ, and bisect it in R, and on HQ let a circle be described, to which apply AV the side of a square = the difference between the sum of the squares of HA, FQ and YZ, CO. Join QV, draw AF || thereto, and AC making the $\angle CAF = \angle OAH$, and AEF, ABC will be the lines required to be drawn.

DEMONSTRATION. Join HV, cutting AF in L, and draw the other lines as in the first case. Then $CD^2 = CO^2 - OD^2$ and $FG^2 = FQ^2 - QG^2$, therefore $FG^2 - CD^2 = OD^2 - QG^2 + FQ^2 - CO^2$. But $\angle OAH = \angle CAF$, by constr. from which take away the common $\angle OAF$, and the $\angle OAD = HAH$, therf. $HL = OD$, because $HA = AO$, by the constr. and by the semicircle, the angle HVQ, or HLA is a right angle, therefore $QG = VL$. Therefore, $FG^2 - CD^2 = HL^2 - LV^2 + FQ^2 - CO^2 = HA^2 - AV^2 + QF^2 - CO^2$, hence

$$AV^2 = HA^2 + FQ^2 - CO^2 - FG^2 + CD^2, \text{ but by constr.}$$

$$AV^2 = HA^2 + FQ^2 - CO^2 - YZ^2; \text{ therefore}$$

$$FG^2 - CD^2 = YZ^2, \text{ and } 4FG^2 - 4CD^2 = EF^2 - BC^2 = 4YZ^2, \text{ equal to the given space by the construction. } Q. E. D.$$

In this case the *maximum* obtains when AV and AL coincide, and the *minimum* when AV = AH.

XXVIII. Or, PRIZE QUESTION 210, answered by the Proposer, Mr. Wm. Wallace, Perth Academy.

Fig. 493, Pl. 26. Suppose PABC, &c. to be drawn as required.

Let the circle PA cut the remaining circles in the points R, S, &c. Draw PH touching the circle PBR and meeting the circle PAR in H. Join HA, HP, HR, RA, RB. Because
RP

PR is a chord in the circle PBR, and PH a tangent at the point P, the angle RBP = RPH = RAH, now the angle RPB = RHA, therefore the triangles RPB, RHA are similar, hence $HR : PR :: HA : PB$, now PH, being a tangent to a given circle at a given point, is given by position, therefore the point H is given, and since the point R is given, the lines HR, PR are both given in magnitude, therefore the ratio of HA to PB, and consequently of HA^2 to PB^2 is given, and $PB^2 = (PR^2 \div HR^2) \cdot HA^2$. In like manner if PK be drawn touching the circle PCS, and meeting the circle PARS at K, and KA, KS, SP be joined, it may be shewn that the point K is given, and also the ratio of KA^2 to PC^2 , or $PC^2 = (PS^2 \div KS^2) \cdot KA^2$, and so on if there be more circles passing through the point P, therefore

$$PA^2 + \frac{PR^2}{HR^2} \cdot HA^2 + \frac{PS^2}{KS^2} \cdot KA^2 + \&c. = PA^2 + PB^2 +$$

$PC^2 + \&c. = Q$ a given space (by hypothesis). Hence it appears that the locus of the point A is the circumference of a given circle. (*Apolon. Loci Plani, Lib. II. Prop. 5, also Stewart's Gen. Theor. sec Math. Rep. Vol. I. page 248.*) Now A is also in the circumference of the given circle PRS, therefore the point A is given, and may be found by this Construction.

From the given point P draw straight lines PH, PK, &c. touching all the circles except one, and meeting the remaining circle in the points H, K, &c. Let the circle to which PH is a tangent meet the circle HKP in R. Join HR, PR, and conceive weights placed at P, and H proportional to 1 and $(PR^2 \div HR^2)$, or to a and b , putting $a = 1$ and $b = (PR^2 \div HR^2)$. Let the circle to which PK is a tangent meet the circle HKP in S, join KS, PS and conceive a weight placed at K proportional to $(PS^2 \div KS^2) = c$, and so on if there be more circles. Find x the centre of gravity of the weights $a, b, c, \&c.$ join PX, HX, KX, &c. and

on X as a centre, with radius equal to $\sqrt{\left(\frac{1}{a+b+c+\&c.} \left\{ Q - a \cdot PX^2 - b \cdot HX^2 - c \cdot KX^2 \&c. \right\} \right)}$ describe a cir-

cle MN meeting the circle PKH in A. Join PA cutting the remaining circles in B, C, &c. the line PABC shall be drawn as required. The Synthetic Demonstration will readily follow from the preceding analysis and from Dr. Stewart's 7th General Theor. as demonstrated in the 1st Vol. of the Repository, pa. 249.

Remarks 1. If it had been required to draw PBC so that $a \cdot PA^2 + b \cdot PB^2 + c \cdot PC^2 \&c.$ might be equal to a given space, the

letters a, b, c , &c. denoting given quantities either positive or negative, it is evident from the preceding analysis that the locus of the point A would still have been a circle.

2. The preceding analysis suggests a number of curious propositions, in geometry, for example: Let any three circles given by position intersect each other at the same point P, from P draw any line whatever meeting all the circles in the points A, B, and C: Then $a \cdot PA + b \cdot PB = c \cdot PC$, where a, b , and c represent three invariable lines.

The same, answered by Mr. John Lowry.

ANALYSIS. Fig. 489, Pl. 26. Let the circles PB, PC, &c. intersect the circle PA at E, F, &c. and join EA, FA, &c. EB, FC, &c. PE, PF, &c.

and draw PP', PP' &c. perp. to AE, AF, &c. Then, Eu. II. 4, $PB^2 = PA^2 + 2PA \cdot AB + AB^2$, $PC^2 = PA^2 + 2PA \cdot AC + AC^2$, and so on; \therefore

$$AP^2 + PB^2 + PC^2 \&c. \text{ is } = n \cdot AP^2 + AB^2 + AC^2, \&c. + 2AP \cdot (AB + AC, \&c.)$$

where n represents the number of given circles. But because the points P, E, are given, the angles PAE, PBE, are given, therefore the triangle EAB is given in species; and in the same way it may be shewn that the triangle FAC is given in species; wherefore the ratios of AB to AE, AC to AF, and so on, are given. Let AB be to AE as a to b , AC be to AF as a to c , and so on; Then

$$AB = \frac{a}{b} AE, AC = \frac{a}{c} AF, AB^2 = \frac{a^2}{b^2} AE^2, AC^2 = \frac{a^2}{c^2} AF^2,$$

and so on; \therefore

$$AP^2 + BP^2 + PC^2 \&c. = n \cdot AP^2 + \frac{a^2}{b^2} AE^2 + \frac{a^2}{c^2} AF^2 \&c. + 2AP \left(\frac{a}{b} AE + \frac{a}{c} AF \&c. \right).$$

Now, by Euc. II. 13, $2AP' \cdot AE + PE^2 = AP^2 + AE^2$, $2AP' \cdot AF + PF^2 = AP^2 + AF^2$, and so on, but, because the angles PAP', PAP' &c. are given, the ratios of PA to AP', PA to AP' &c. are given. Let PA be to AP' as d to a , PA to AP' as e to a &c. Then

$$2AP \cdot AE : 2AP' \cdot AE = AP^2 + AE^2 - PE^2 :: d : a, \text{ therefore}$$

$$2AP \cdot \frac{a}{b} \cdot AE = \frac{d}{b} \cdot (AP^2 + AE^2 - PE^2).$$

In the same way it is shewn that,

$2AP \cdot \frac{a}{c} \cdot AF$ is $= \frac{c}{c} \cdot (AP^2 + AF^2 - PF^2)$, and so on, that is,

$2AP \cdot \frac{a}{b} \cdot AE = \frac{d}{b} \cdot PA^2 + \frac{db}{b^2} \cdot AE^2 - \frac{d}{b} \cdot PE^2$, and

$2AP \cdot \frac{a}{c} \cdot AF = \frac{c}{c} \cdot PA^2 + \frac{cc}{c^2} \cdot AF^2 - \frac{c}{c} \cdot PF^2$. Therefore, it is

evident that, $AP^2 + BP^2 + CP^2$ &c. is equal to the excess of

$(n + \frac{d}{b} + \frac{c}{c} \&c.) \cdot AP^2 + (\frac{a^2 + db}{b^2} \&c.) \cdot AE^2 + (\frac{a^2 + bc}{c^2} \&c.)$

AF^2 &c. above $\frac{d}{b} \cdot PE^2 + \frac{c}{c} \cdot PF^2$ &c. But PE and PF &c. are

given lines, therefore, it is evident the sum of the spaces to which the squares of AP , AE , AF , &c. have given ratios, is equal to a given space. Wherefore, by Prop. D, Art. XXXI. Vol. I. of the Repository, the *locus* of the point A is a circle given by position, and therefore the method of construction is evident from that Proposition.

From the above Analysis the following Theorems are evident.

I. If the sum of the solids contained under the squares of the chords, and given lines, be equal to a given solid, the *locus* of the point A will be a circle given by position.

II. If there be any number of given points, P , Q , R , S , &c. and a straight line be drawn through P , so that the sum of the squares of all the perpendiculars QA , RB , SC , &c. may be equal to a given space; or that the sum of the solids contained by these squares and given lines, may be equal to a given solid; the *locus* of the point A will in both cases be a circle given by position.

III. If the difference between the sum of the squares (or the spaces to which they have given ratios) of any assigned number of the chords, and the sum of the squares (or the spaces to which they have given ratios) of the remaining chords be equal to a given space; the *locus* of the point A will still be a circle given by position.

IV. If PEF be a given circle, and PF , PE , &c. be chords given by position, and if from a point A , in the circumference, the perpendiculars Ap , Ap' , &c. be drawn to the chords PE , PF , &c. and AG , AH , AI , &c. be drawn to the given points G , H , I , &c. so that the sum of the squares of GA , HA , IA , &c. (or the spaces to which they have given ratios) together with the rectangles contained by Ap , Ap' , &c. and given lines, may be equal

to a given space; the *locus* of the point A will be a circle given by position.

It is also evident from the solutions to questions 180 and 209, and from what is done above, that solutions to the following problems may be readily had.

I. Let P be a given point, and AB, CD, EF, &c. be circles given in magnitude and position: It is required to draw a straight line, through P, intersecting the circles in the points AB, CD, EF, &c. so that the sum of the squares (or the spaces to which they have given ratios,) of all the chords AB, CD, EF, &c. may be equal to a given space.

II. Let P be a given point and AB, CD, EF, &c. and *ab*, *cd*, *ef*, &c. be circles given in magnitude and position: It is required to draw two straight lines, through P, intersecting the circles in the points AB, CD, EF, &c. and *ab*, *cd*, *ef*, &c. to contain a given angle at P, and so that the sum of the squares of all the chords AB, CD, EF, &c. *ab*, *cd*, *ef*, &c. (or the spaces to which they have given ratios) may be equal to a given space.

The same, answered by Mr. Richard Nicholson, Private Teacher of the Mathematics, at Liverpool.

ANALYSIS. Fig. 490, pl. 26. Join ZA, Z being one of the points of intersection of the circles of which PA and PB are the chords, and produce ZA to meet the circle of which PB is the chord at Y, and join PY. Then the angles PAZ ($= PYA + APY$) and PYA are given, therefore the angle APY, and the arch BY are given, and therefore the triangle PYA is given in species; wherefore $PA^2 : PY^2$ has a given ratio, which suppose as $a : b$; then $b \cdot PA^2 = a \cdot PY^2$, therefore $a \cdot PY^2 + b \cdot PB^2 = b \cdot PA^2 + b \cdot PB^2 =$ a given solid, when the number of circles is two, therefore the chord PB is given, by the Lemma on pa. 174, Simpson's Select Exercises.

Join PX, X being one of the points of intersection of the circles of which PB and PC are the chords, and upon PX, as a diameter, let a circle be described meeting XY, CX, and XB in W, U, and V. Join also PW, PU, and PV. Then since the angles PYX, PCX, and PBX, of the right angled triangles PYW, PCU, and PBV are given, as also the angle PBY, the arches VW, UW will be given, and therefore these triangles will be given in specie; wherefore $PY^2 : PW^2$ has a given ratio, which suppose as $c : a$, and $PB^2 : PV^2$ has a given ratio, which suppose as $d : b$, and $PC^2 : PU^2$ has also a given ratio, which suppose as $e : b$. Then

$a \cdot PY^2 = c \cdot PW^2$, $b \cdot PB^2 = d \cdot PV^2$, and $b \cdot PC^2 = e \cdot PU^2$, therefore $c \cdot PW^2 + d \cdot PV^2 + e \cdot PU^2 = b \cdot PA^2 + b \cdot PB^2 + b \cdot PC^2 =$ a given solid

Q R Q

Solid when the number of circles is three, and therefore *PU* is given by the problem immediately following the above-mentioned lemma.

Join *PT*, *T* being one of the points of intersection of the circles of which *PD* is the chord of one, and *PX* the diameter of the other. Upon the diameter *PT* let a circle be described, meeting *TW*, *TU*, *TV*, and *TD* in *S*, *Q*, *R*, and *O*. Join *PS*, *PQ*, *PR*, and *PO*. Then by proceeding as above, we have $f \cdot PS^2 + g \cdot PR^2 + h \cdot PQ^2 + i \cdot PO^2 = b \cdot PA^2 + b \cdot PB^2 + b \cdot PC^2 + b \cdot PD^2 =$ a given solid, when the number of circles is four, and the arches *RS*, *RQ*, and *QO* being given, *PO* will be given by the problem referred to above. And by thus proceeding may the problem be solved for any number of circles.

The same, answered by Mr. J. H. Swale.

ANALYSIS. Fig 491, Pl. 26. Suppose the line *PAB*, &c. to be drawn as required by the problem. Through *P* let the diameters *PF*, *PG*, *PH*, &c. be drawn, and join *FA*, *GB*, *HC*, &c. and they will evidently be parallel to each other. In these diameters let *PL*, *PM*, &c. be taken, each equal to the diameter *PF*, and demit, upon *PABC*, &c. the perpendiculars *LQ*, *MR*, &c. Then

$PB^2 : PQ^2 = PG^2 : PL^2$ (PF^2), and $PC^2 : PR^2 = PH^2 : PM^2$ (PF^2), and so on.

That is, PB^2 is = to the space to which PQ^2 has the same ratio that PG^2 has to PF^2 , and PC^2 is = to the space to which PR^2 has the same ratio that PH^2 has to PF^2 , and so on.

Now, perpendicular to *PG*, *PH*, &c. take *GA'*, *HC'*, &c. each = to *PF*, and join *P* with the points *A'*, *C'*, &c. and perpendicular to *PA'*, *PC'*, &c. draw *PB'*, *PD'*, &c. meeting *A'G*, *C'H*, &c. produced in *B'*, *D'*, &c. Then $PG^2 : A'G^2 (PF^2) = GB' \cdot GA'$, and $PH^2 : C'H^2 (PF^2) = HD' \cdot HC'$, and so on.

That is, the space to which PQ^2 has the same ratio that PF has to GB' is = PB^2 , and the space to which PR^2 has the same ratio that PF has to HD' is = PC^2 , and so on. Therefore the sum of the squares of *PA*, *PB*, *PC*, &c. is = to the square of *PA*, together with the space to which the square of *BQ* has the same ratio that PF has to GB' together with the space to which the square of *PR* has the same ratio that PF has to HD' , and so on. But *PF*, GB' , HD' , &c. are given magnitudes, and *F*, *L*, *M*, &c. are given points, through which the parallels *FA*, *LQ*, *MR*, &c. are to pass. Whence, by the Demonstrations to Stewart's General Theorems in the Repository, a point *U* may always be found such, that a parallel *UW* being drawn through *U*, and the perpendicular

dicular PV demitted thereon from P; the square of PA, together with the space to which the square of PQ has the same ratio that PF has to GB', together with the space to which the square of PR has the same ratio that PF has to HD', and so on, shall be equal to the space to which the square of PV has the same ratio that $\frac{1}{2}$ PF has to the sum of PF, HP', HD', &c. equal, in the present case, to a given space.

Let U be thus determined for the given points P, F, L, M, &c. and the given magnitudes PF, GB', HD' &c. Join PU and produce it U', making $PU^2 : PU'^2 = \frac{1}{2}PF : PF + GB' + HD',$ &c. and draw U'V'W' parallel to UVW.

Then, by parallels, $UP^2 : U'P^2 = PV^2 : V'P^2 = \frac{1}{2}PF + GB' + HD' \&c.$ that is, $V'P^2$ is equal to a given space. Therefore it only remains to describe a semicircle upon PU', apply therein PV the side of a square = to the given space, and through the points V', P draw the line required meeting the circles in A, B, C, &c.

Hence the solutions of the following Problems are evident.

Prob. I. To draw a straight line through a given point P such, that if perpendiculars be demitted thereon from any number of given points, the sum of the squares of these perpendiculars shall be equal to a given space.

Prob. II. Let there be any number of given points in the circumference of a circle given in magnitude and position: It is required to draw a diameter therein such, that the sum of the squares of the parallel chords, drawn perpendicular to that diameter through the given points, shall be equal to a given space.

Mr. Richard Nicholson is requested to send to Mr. Glendinning's for the Medal for solving the Prize Question.

ARTICLE LI.

MATHEMATICAL QUESTIONS.

To be answered in Number XLII.

I. QUESTION 241, by Mr. Tho. Bulmer, Sunderland.

TWO vessels of equal dimensions in all respects, and in form of the frustum of a cone, were placed perpendicularly to the horizon,

horizon, the one with its greater end up and the other with its less end up, and the liquor with which they were filled exactly even with the surface of the earth.—Now, although the diameters and depths of the two vessels were exactly the same, yet, it was found that the vessel which stood on its less end, would hold more than the other by 10 ale gallons. Required the diameters and contents of the two vessels, supposing the depth of each 12 feet, the greater diameter to the less as 4 to 1, and the earth a perfect sphere whose radius is 4000 miles?

II. QUESTION 242, *by Mr. William Marrat, Boston.*

From a cone, the length of which was $7\frac{1}{2}$ feet and the diameter of its base 26 inches, I ordered 4 slices to be cut off, by sections parallel to the axis, and at such a distance as just to reduce the base to a square. Query the solidity and superficies of the part remaining?

III: QUESTION 243, *by Mr. Wm. Peacock, Birmingham.*

A land surveyor, measuring a triangular piece of ground, found the distances of the three points where perpendiculars from the angles met the opposite sides to be 9, 10 and 11 chains. From these data to determine the area of the field by trigonometry.

IV: QUESTION 244, *by Merones Minor.*

To determine the thickness of a triangular wall at the bottom, necessary to support a body of water; that side of the wall facing the water being upright, the depth of the water 10 feet, and the height of the wall 12 feet; the specific gravity of the wall and water being as 11 to 7.

V. QUESTION 245, *by Mr. Olinthus Gilbert Gregory, Teacher of Mathematics, Cambridge.*

A carpenter has received orders to make an elliptical table the transverse diameter of which is to be 12 feet; the length of the conjugate he has forgotten; but he knows the ellipsis was to be such that the difference between the areas of the greatest inscribed triangle and of the least circumscribing triangle, was to be $70\frac{1}{4}148025$ feet. From this he hopes some ingenious gentleman will tell him the length of the conjugate diameter.

VI.

VI. QUESTION 246, by Mr. Newton Bosworth, *Cambridge*.

There are two plane mirrors, each 3 feet 6 inches long, and, meeting each other at one end, form an angle of 15° . Now, if a small object be placed 6 inches from the other extremity of one of these mirrors, it is required to determine how many images of the object will be formed upon the mirrors by reflection, and how far each image is behind the surface of the mirror upon which it is formed.

VII. QUESTION 247, by Mr. O. G. Gregory, *Cambridge*.

A gentleman who has a garden in form of a regular pentagon each side of which measures 50 yards, is desirous of knowing without actual admeasurement the sum of the lengths of walks which shall be made to connect each angle with its two opposite ones; and wishes also to know the expence of a pond which shall be dug in the form of the most regular curve that can be made to touch all the walks, reckoning the expence of digging the pond at 6d. per yard on the surface.

VIII. QUESTION 248, by Urfa Miner,

What is the nature of that curve, the distance of the sub-tangent of which from its vertex is equal to the ordinate at the point of contact of the tangent?

IX. QUESTION 249, by Mr. I. T. M'Donell.

A, B, and C are three given points in the circumference of a circle: it is required to draw a chord CD, in the circle, such, that, being divided in a given ratio at P, the sum or difference of the squares on AP and BP, may be equal to a given space, and to shew the limits.

X. QUESTION 250, by Hypatia.

Given the area and the perimeter of a regular polygon to determine the number of its sides.

XI. QUESTION 251, by Mr. T. S. Evans, *Mathematical Master at Christ's Hospital*.

At what height above the surface of a table must the flame of a candle be placed so as to illuminate, in the greatest degree possible, a point of the table at a given distance from the point directly under the candle?

XII.

XII. QUESTION 252, *by Mr. Henry Boyley.*

Given the circumference of a cylinder standing on its base perpendicular to the horizon $= a$, the diameter of a cord wound in a spiral upon it $= b$, and its length $= l$; also there is given the angle the rounds form with the horizon; to find the length of the curve made by the end of the cord in winding it all off, the cord being always stretched and its end continually depressing.

XIII. QUESTION 253, *by Mr. Louis Hill.*

There is a perfectly polished circular table, whose diameter is 20 feet, placed so that its plane makes an angle of 10 degrees with the horizon. Now if a ball be projected from a point in the periphery, which is distant from the highest point 10 inches, and with a velocity of 8 feet in a second, it is required to determine the direction of the projectile, and the point where the ball will quit the table, and the length of the track, supposing it to pass through the centre of the table.

XIV. QUESTION 254, *by Collator.*

Two equal fires are placed in the foci of an ellipsis; a person walks from one apsis to the other along the periphery; where will he feel the least heat?

XV. QUESTION 255, *by Mr. J. H. Swale.*

There are two given circles AEBR, CHDI, whose centres are C and O; the latter passing through the centre C of the former. It is required to draw the chord CP, such that, demitting a perpendicular thereon from a given point G, the part EP intercepted between the circles shall be bisected by that perpendicular.

XVI. QUESTION 256, *by Mr. James Cunliffe, Bolton.*

Theorem. ABC is a plane Δ circumscribed by a circle, the base AB being bisected in G by the diameter FE, and CH drawn parallel to AB meeting FE in H: if upon GH as a diameter another circle is described, and through O the centre of the circumscribed circle a perpendicular to FE be drawn meeting the circumference of the circle, whose diameter is GH, in R, and the line FR be drawn from the point F, below the base AB: then will $4 \times FR^2 = AC^2 + BC^2$. Required the demonstration?

XVII.

XVII. QUESTION 257, *by Amicus.*

Suppose a straight rod to be partly immersed in a vessel of water ; to determine the angle at which it must be inclined to the surface that the apparent bending at the surface may be a maximum.

XVIII. QUESTION 258, *by Amicus.*

The latitude being given, to find at what time, on the longest day, the variation of the sun's altitude is greatest.

XIX. QUESTION 259, *by A Miner.*

In order to discover the position of a stratum of coal three points A, B, C were assumed on the earth's surface, the distances AB, AC, BC were found to be a , b , and c fathoms, and the line AB made with the meridian an angle of g degrees. After boring at the points A, B, C, the depth of the coal below the surface was found to be p , q and r fathoms respectively. From these data it is required to find the direction and dip of the stratum.

XX. QUESTION 260, *by Collator:*

The axis of a given cylinder passes through the centre of a system whose weight is known ; and the whole is made to revolve by a given weight P, suspended by a string wound round the cylinder ; the space, then, through which P descends in a given time, being known, it is proposed to find the centre of gyration of the whole system.

XXI. QUESTION 261, *by Mr. James Cunliffe.*

To determine the greatest ordinate of a catenarian curve of a given length, when the solid formed by its revolution about that ordinate is a maximum.

XXII. QUESTION 262, *by the Rev. L. Evans.*

Suppose two weights connected by a string, which passes over a pulley, and that one of the weights is placed upon an inclined plane, while the other hangs in the air by the connecting string. It is required to determine the *locus* of the centre of gravity of the weights, supposing the relative position of the pulley and plane, the length of the string, and the weights to be all given.

XXIII. QUESTION 263, *by Collator.*

Suppose a perfectly polished tube to pass through the centre of the earth in the plane of the equator; it is proposed to find the nature of the spiral described by a body descending in this tube by its own weight, while it is carried about at the same time by the earth's rotation.

XXIV. QUESTION 264, *by Mr. J. H. Swale.*

There are two given points in the periphery of a given circle, It is required to apply in the circle a chord of a given length such, that demitting perpendiculars thereon from the given points, the ratio, sum, difference, rectangle, difference of the squares or sum of the squares of those perpendiculars shall be given.

XXV. QUESTION 265, *by Mr. Cunliffe.*

Let one end of a perfectly flexible heavy line or chain APB of a given length be fastened at the point A whilst the other end B is carried along the horizontal line AB. It is required to find the equation and quadrature of the curve which is the *locus* of a given point P in the chain.

XXVI. QUESTION 266, *by Mr. John Johnson, Birmingham.*

There are two circles given in magnitude and position, and a point in the straight line joining their centres is also given. It is required to describe another circle, so that its circumference may pass through that point, and touch one of the given circles, and have its centre in the circumference of the other given circle.

XXVII. QUESTION 267, *by Mr. John Lowry.*

Given the base of a plane triangle to construct it when the sum of the squares of the sides has to the area of the triangle a given ratio, and when the greatest side has to the less, the greatest ratio possible.

XXVIII. QUESTION 268, *by Mr. Lowry.*

Given the vertical angle, and the difference between the square of the base and the rectangle under the sum of the sides and a given line, to construct the Δ , when the perpendicular from one of the angles at the base upon the opposite side is a maximum.

XXIX.

XXIX. QUESTION 269, by Mr. Lowry.

In a circle given in magnitude and position, it is required to inscribe a trapezium such that three of its sides shall pass through three given points in the same straight line, and that the fourth side shall touch a circle given in magnitude and position.

XXX. PRIZE QUESTION 270, by Philalæthes Cantabrigienfis.

Given $\frac{x}{(1 + \sqrt{x + 2\sqrt{x}})^2} = y$, in which equation x and y begin together, to find the value of y when $x = \frac{9}{16}$.

ARTICLE LII.

EIGHT PROPOSITIONS FROM LAWSON.

(To be answered in Number XIII.)

PROP. XLVII.

IF from one angle A of a rectangle ABCD a line be drawn to cut the two opposite sides BC, DC, the former in F, and the latter produced in E; then I say that the rectangle EAF is equal to the sum of the rectangles EDC, CBF.

PROP. XLVIII.

If a rectangle be inscribed in a right-angled triangle, so that one of its angles coincide with the angle of the triangle; then I say that the rectangle under the segments of the hypothenuse is equal to the sum of the rectangles under the segments of the sides about the right angle made by this inscription.

PROP. XLIX.

If from the same point C two tangents be drawn to a semicircle whose diameter is AB, and if the extremes of the diameter and the points of contact be joined, either cross-ways by two lines intersecting in F, or other-ways by two lines intersecting in H; then I say that CF or HC produced to meet the diameter AB will be perpendicular to the same.

PROP

PROP. I.

If in a semicircle whose diameter is AB the chord of 60° , equal to the radius be inscribed, and from the centre E a perpendicular drawn thereto and produced to meet the circumference in F; then I say that AF, EF, BF are continual proportionals.

PROP. LI.

If a line be cut in extreme and mean proportion; then I say that the square of the whole, the rectangle under the whole and the greater segment, and the rectangle under the whole and the lesser segment are continual proportionals.

PROP. LII.

If three lines are continual proportionals; the sum of the squares of the mean and the greater extreme is to the rectangle contained under the same, as the sum of the extremes is to the mean.

PROP. LIII.

In every right-angled triangle, as the hypotenuse is to the sum of the sides about the right angle, so is the said sum, to the sum of the hypotenuse and twice the perpendicular from the right-angle.

PROP. LIV.

If a right line AD be any ways cut in B, and from thence a perpendicular BE be erected equal to a mean proportional between the whole AD and the part AB, and a circle be drawn through the points A, D, E, and from A a perpendicular be erected to meet the circumference in F; then I say AF, AB, BE, AD are four continual proportionals.

END OF THE SECOND VOLUME.

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